ARITHMETIC PROGRESSIONS

Concepts Covered

- Arithmetic Progressions-Introduction
- nth term of AP
- Sum of the first n terms of an AP

Introduction

Consider the following arrangement of numbers:
(i) 1, 3, 5, 7, …….. (ii) 3, 6, 12, 24, …….. (iii) 1, 4, 9, 16, ……..
In each of the above arrangements, we observe some patterns. In (i) we find that the succeeding terms are obtained by adding a fixed number [i.e. 2], in (ii) by multiplying with a fixed number [i.e. 2], in (iii) we find that they are squares of natural numbers.
In this chapter, we shall discuss one of these patterns in which succeeding terms are obtained by adding a fixed number to the preceding terms. We shall also see how to find their n^{th} terms and the sum of n consecutive terms, and use this knowledge in solving some of our daily life problems.

Historical Facts

Gauss was a very talented and gifted mathematician of 19^{th} century who developed the formula:
$1 + 2 + 3 + 4 + \ldots + (n - 1) + n = \frac{n(n+1)}{2}$ for the sum of first n natural numbers at the age of 10. He did this in the following way:
$S = 1 + 2 + 3 \ldots + (n - 2) + (n - 1) + n$
$S = \underline{n + (n - 1) + (n - 2) + \ldots + 3 + 2 + 1}$
$2S = (n + 1) + (n + 1) + (n + 1) + \ldots + (n + 1) + (n + 1) + (n + 1)$
$= (n + 1)(1 + 1 + 1 + \ldots$ up to n times)
$2S = (n + 1)n \Rightarrow S = \frac{n(n+1)}{2}$

Sequence

Even when Gauss was a little child of three he could read and make mathematical calculation himself. Gauss proved the fundamental theorem of Algebra when he was 20 years old. His contribution to mathematics has been immense because his formulae were used in applied field of Astronomy, Differential Geometry and Electricity widely all over the world by scientists.
In our daily life, we come across the arrangement of numbers or objects in an order such as arrangement of students in a row as per their roll numbers, arrangement of books in the library, etc.
An arrangement of numbers depends on the given rule:

Given Rule	Arrangement of numbers
Write 3 and then add 4 successively	3, 7, 11, 15, 19,….
Write 3 and then multiply 4 successively	3, 12, 48, 192,…..
Write 4 and then subtract 3 successively	4, 1, - 2, -5,…..
Write alternately 5 and – 5	5, - 5, 5, -5,…

Thus, a sequence is an ordered arrangement of numbers according to a given rule.
Terms of a Sequence: The individual numbers that form a sequence are the terms of a sequence.
For example: 2, 4, 6, 8, 10,…… forming a sequence are called the first, second third, fourth and fifth,…. terms of the sequence.
The terms of a sequence in successive order is denoted by T_n or a_n The nth term T_n is called the general term of the sequence.

Example:

Write the first five terms of the sequence, whose nth term is $a_n = \{1 + (-1)^n\}n$.

Solution: $a_n = \{1 + (-1)^n\}n$

Substituting n = 1, 2, 3, 4 and 5, we get

$a_1 = \{1 + (-1)^1\}\ 1 = 0$; $a_2 = \{1 + (-1)^2\}\ 2 = 4$;

$a_3 = \{1 + (-1)^3\}\ 3 = 0$; $a_4 = \{1 + (-1)^4\}\ 4 = 8$;

$a_5 = \{1 + (-1)^5\}\ 5 = 0$

Thus, the required terms are: 0, 4, 0, 8 and 0.

Example:

Find the 20th term of the sequence whose nth term is, $a_n = \frac{n\,(n+2)}{n+3}$

Solution: $a_n = \frac{n\,(n+2)}{n+3}$. Putting n = 20, we obtain $a_{20} = \frac{20(20-2)}{20+3}$

Thus, $a_{20} = \frac{360}{23}$

Example:

Write the first five terms of the sequence defined by $a_n = (-1)^{n-1} \cdot 2^n$

Solution: $a_n = (-1)^{n-1} \times 2^n$

Putting n = 1, 2, 3, 4, and 5 we get

$a_1 = (-1)^{1-1} \times 2^1 = (-1)^0 \times 2 = 2$

$a_2 = (-1)^{2-1} \times 2^2 = (-1)^1 \times 4 = -4$

$a_3 = (-1)^{3-1} \times 2^3 = (-1)^2 \times 8 = 8$

$a_4 = (-1)^{4-1} \times 2^4 = (-1)^3 \times 16 = -16$

$a_5 = (-1)^{5-1} \times 2^5 = (-1)^4 \times 32 = 32$

Thus, the first five term of the sequence are 2, –4, 8, –16, 32.

Progression

It is not always possible to write each and every sequence of some rule

For example of prime numbers 2, 3, 5, 7, 11,… cannot be expressed explicitly by stating a rule and we do not have any expression for writing the general term of this sequence.

The sequence that follows a certain pattern is called a progression. Thus, the sequence 2, 3, 5, 7, 11,… is not a progression. In a progression, we can always write the nth term.

Consider the following collection of numbers : (i) 1, 3, 5, 7… (ii) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ ….

Arithmetic Progressions

An arithmetic progression is that list of numbers in which the first term is given and each term, other than the first term is obtained by adding a fixed number ‘d’ to the preceding term.

The fixed term ‘d’ is known as the common difference of the arithmetic progression. It’s value can be positive, negative or zero. The first term is denoted by ‘a’ or ‘a_1’ and the last term by 'ℓ'.

Ex. Consider a sequence 6, 10, 14, 18, 22, …

Hence, $a_1 = 6$, $a_2 = 10$, $a_3 = 14$, $a_4 = 18$, $a_5 = 22$

$a_2 - a_1 = 10 - 6 = 4$

$a_3 - a_2 = 14 - 10 = 4$

$a_4 - a_3 = 18 - 14 = 4$

Therefore, the sequence is an arithmetic progression in which the first term a = 6 and the common difference d = 4.

Example:

Write first four terms of the AP, when the first term a and the common difference d are given as follows

(i) a = 4, d = 5 **(ii) a = - 1.25, d = - 0.25**

Solution: (i) a = 4, d = 5

First term, a = 4

Second term = 4 + d = 4 + 5 = 9

Third term = 9 + d = 9 + 5 = 14

Fourth term = 14 + d = 14 + 15 = 19

Hence, first four terms of the given AP are 4, 9, 14, 19.

(ii) $a = -1.25, d = -0.25$

First term = a = – 1.25

Second term = – 1.25 + d = – 1.25 + (– 0.25) = – 1.50

Third term = – 1.50 + d = – 1.50 + (–0.25) = – 1.75

Fourth term = - 1.75 + d = - 1.75 + (-0.25) = – 2.00

Hence, first four terms of the given AP are – 1.25, – 1.50, – 1.75, – 2.00

Example:

For the AP $\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, \ldots\ldots$ write the first term a and the common difference d. Also write the next two terms after the given last term $-\frac{3}{2}$.

Solution: We have $a_1 = \frac{3}{2}, a_2 = \frac{1}{2}, a_3 = -\frac{1}{2}, a_4 = -\frac{3}{2}$ and so on.

Thus, $a = \frac{3}{2}$

$a_2 = a_1 = \left(\frac{1}{2}\right) - \left(\frac{3}{2}\right) = -1,$

$a_3 - a_2 = \left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right) = -1,$

$a_4 - a_3 = \left(-\frac{3}{2}\right) - \left(-\frac{1}{2}\right) = -1$, and so on.

$\Rightarrow d = -1$

Now, we find the successor of $-\frac{3}{2}$.

$a_5 = \left(-\frac{3}{2}\right) + d = \left(-\frac{3}{2}\right) + (-1) = -\frac{5}{2}$

Then $a_6 = a_5 + d = \left(-\frac{5}{2}\right) + (-1) = -\frac{7}{2}$

Hence, the next two terms after the given term $-\frac{3}{2}$ are $-\frac{5}{2}, -\frac{7}{2}$.

General Form of an A.P. :

If we denote the starting number i.e. the 1st number by 'a' and a fixed common difference by 'd' then a, a + d, a + 2d, a + 3d, a + 4d, ... forms an A.P.

n^{th} term of an A.P. :

Let A.P. be a, a + d, a + 2d, a + 3d, ...

Then,

First term (a_1) = a + 0.d

Second term (a_2) = a + 1.d

Third term (a_3) = a + 2.d

.   .

.   .

.   .

n^{th} term (a_n) = a + (n - 1) d

∴ a_n = a + (n - 1) d is called the n^{th} term.

Example:

Find the $12^{th}, 24^{th}$ and nth term of the A.P. given by $9, 13, 17, 21, 25, \ldots\ldots\ldots$

Solution: We have, First term, $a = 9$ and, Common difference, $d = 4$

$[\because 13 - 9 = 4, 17 - 13 = 4, 21 - 7 = 4$ etc. $]$

We know that the nth term of an A.P. with first term a and common difference d is given by

$a_n = a + (n - 1)d$

Therefore, $a_{12} = a + (12 - 1)d$

$= a + 11d = 9 + 11 \times 4 = 53$

$a_{24} = a + (24 - 1)d$

$= a + 23d = 9 + 23 \times 4 = 101$

and, $a_n = a + (n - 1)d$

$= 9 + (n - 1) \times 4 = 4n + 5$

$a_{12} = 53, a_{24} = 101$ and $a_n = 4n + 5$

Example:

Which term of the sequence $-1, 3, 7, 11, \ldots,$ is 95 ?

Solution: Clearly, the given sequence is an A.P.

We have, $a =$ first term $= -1$ and, $d =$ Common difference $= 4$.

Let 95 be the n^{th} term of the given A.P. then, $a_n = 95$

$\Rightarrow a + (n-1)d = 95$

$\Rightarrow -1 + (n-1) \times 4 = 95$

$\Rightarrow -1 + 4n - 4 = 95 \Rightarrow 4n - 5 = 95$

$\Rightarrow 4n = 100 \Rightarrow n = 25$

Thus, 95 is 25^{th} term of the given sequence.

Example:

Which term of the sequence $4, 9, 14, 19, \ldots$ is 124 ?

Clearly, the given sequence is an A.P. with first term $a = 4$ and common difference $d = 5$.

Solution: Let 124 be the n^{th} term of the given sequence. Then, $a_n = 124$

$a + (n-1)d = 124$

$\Rightarrow 4 + (n-1) \times 5 = 124 \Rightarrow n = 25$

Hence, 25^{th} term of the given sequence is 124 .

Example:

The 10^{th} term of an A.P. is 52 and 16^{th} term is 82 . Find the 32^{nd} term and the general term.

Solution: Let a be the first term and d be the common difference of the given A.P. Let the A.P. be $a_1, a_2, a_3, \ldots . a_n, \ldots \ldots$

It is given that $a_{10} = 52$ and $a_{16} = 82$

$\Rightarrow a + (10-1)d = 52$ and $a + (16-1)d = 82$

$\Rightarrow a + 9d = 52$ and, $a + 15d = 82$

Subtracting equation (ii) from equation (i), we get

$-6d = -30 \Rightarrow d = 5$

Putting $d = 5$ in equation (i), we get

$a + 45 = 52 \Rightarrow a = 7$

$\therefore a_{32} = a + (32-1)d = 7 + 31 \times 5 = 162$

and, $a_n = a + (n-1)d = 7(n-1) \times 5 = 5n + 2.$

Hence $a_{32} = 162$ and $a_n = 5n + 2.$

Example:

Determine the general term of an A.P. whose 7^{th} term is -1 and 16^{th} term 17 .

Solution: Let a be the first term and d be the common difference of the given A.P. Let the A.P. be

$a_1, a_2, a_3, \ldots \ldots . a_n, \ldots \ldots ..$

It is given that $a_7 = -1$ and $a_{16} = 17$

$a + (7-1)d = -1$ and, $a + (16-1)d = 17 \Rightarrow a + 6d = -1$ and, $a + 15d = 17$

Subtracting equation (i) from equation (ii),

We get

$9d = 18 \Rightarrow d = 2$

Putting $d = 2$ in equation (i), we get

$a + 12 = -1 \Rightarrow a = -13$

Now, General term $= a_n$

$= a + (n-1)d = -13 + (n-1) \times 2 = 2n - 15$

Now, General term $= a_n$

$= a + (n-1)d = -13 + (n-1) \times 2 = 2n - 15$

Example:

Which term of the sequence $72, 70, 68, 66, \ldots$ is 40 ?

Solution: Here 1^{st} term $x = 72$ and common difference $d = 70 - 72 = -2$

$\therefore$ For finding the value of n

$a_n = a + (n-1)d$

$\Rightarrow 40 = 72 + (n-1)(-2)$

$\Rightarrow 40 - 72 = -2n + 2$

$\Rightarrow -32 = -2n + 2$

$\Rightarrow -34 = -2n$

$\Rightarrow n = 17$

$\therefore 17^{th}$ term is 40

Example:

Is 184, a term of the sequence $3, 7, 11, \ldots$?
Solution: Here 1 st term $(a) = 3$ and common difference $(d) = 7 - 3 = 4$

n^{th} term $(an) = a + (n-1)d$
$\Rightarrow 184 = 3 + (n-1)4$
$\Rightarrow 181 = 4n - 4$
$\Rightarrow 185 = 4n$
$\Rightarrow n = \dfrac{185}{4}$
Since, n is not a natural number.
$\therefore$ 184 is not a term of the given sequence.

Example:

Which term of the sequence $20, 19\frac{1}{2}, 18\frac{1}{2}, 17\frac{3}{4}$ is the 1^{st} negative term?
Solution: Here, 1^{st} term $(a) = 20$, common difference $(d) = 19\frac{1}{4} - 20 = -\frac{3}{4}$

Let n^{th} term of the given A.P. be 1^{st} negative term $\therefore a_n < 0$ i.e. $a + (n-1)d < 0$
$\Rightarrow 20 + (n-1)\left(-\frac{3}{4}\right) < 0 \Rightarrow \frac{83}{4} - \frac{3n}{4} < 0$
$\Rightarrow 3n > 83 \Rightarrow n > \frac{83}{3} \Rightarrow n > 27\frac{2}{3}$
Since, 28 is the natural number just greater then $27\frac{2}{3}$.
$\therefore 1^{st}$ negative term is $28^{th.}$.

Example:

If p^{th}, q^{th} and r^{th} term of an A.P. are a, b, c respectively, then show that $a(q-r) + b(-p) + c(p-q) = 0$.
Solution: Let A be the first term and D be the common difference of the given A.P.

$a_p = a \Rightarrow A + (p-1)D = a$ ... (1)
$a_q = b \Rightarrow A + (q-1)D = b$ ... (2)
$a_r = c \Rightarrow A + (r+1)D = c$ ... (3)
Now,
L.H.S. $= a(q-r) + b(r-p) + c(p-q)$
$= \{A + (p-1)D\}(q-r) + \{A + (q-1)D\}(r-p) + \{A + (r-1)D\}(p-q)$
$= 0$
L.H.S = R.H.S.

Example:

If m times the m^{th} term of an A.P. is equal to n times its n^{th} term. Show that the $(m+n)^{th}$ term of the A.P is zero.
Solution: Let A be the 1^{st} term and D be the common difference of the given A.P.

Then, $ma_m = na_n$
$\Rightarrow m[A + (m-1)D] = n[A + (n-1)D]$
$\Rightarrow A(m-1) + D[m + n(m-n) - (m-n)] = 0$
$\Rightarrow A + (m+n-1)D = 0 \Rightarrow a_{m+n} = 0$

Example:

If the p^{th} term of an A.P. is q and the q^{th} term is p, prove that its n^{th} term is $(p+q-n)$.
Solution: $a_p = q \Rightarrow A + (p-1)D = q$ .... (1)

And,
$a_q = p \Rightarrow A + (q-1)D = p$ .... (2)
By Solving (1) & (2) we get, $D = -1$ and $A = p + q - 1$
$\therefore a_n = A + (n-1)D$
$a_n = (p + q - 1) + (n-1)(-1)$
$a_n = p + q - n.$

Example:

If the m^{th} term of an A.P. is $\frac{1}{n}$ and n^{th} term be $\frac{1}{m}$ then show that its (mn) term is 1.
Solution: $a_m = \frac{1}{n} \Rightarrow A + (m-1)D = \frac{1}{n}$ ... (1)

and, $a_m = \frac{1}{m} \Rightarrow A + (n-1)D = \frac{1}{m}$ ... (2)
By solving (1) & (2), $D = \frac{1}{mn}$ and $A = \frac{1}{mn}$
$\therefore a_{mn} = A + (mn-1)D$

$$= \frac{1}{mn} + (mn - 1)\frac{1}{mn}$$
$$= 1$$

m^{th} Term of an A.P. from the end:

Let **'a'** be the 1^{st} term and **'d'** be the common difference of an **A.P.** having **n** terms. **Then m^{th}** term from the end is **$(n - m + 1)^{th}$** term from beginning or **$\{n - (m - 1)\}^{th}$** term from beginning.

Example:

Find 20^{th} term from the end of an A.P. $3, 7, 11 \ldots\ldots 407$.

Solution: $407 = 3 + (n - 1)4 \Rightarrow n = 102$

$\therefore 20^{th}$ term from end $\Rightarrow m = 20$

$a_{(102-(20-1))} = a_{102-19} = a_{83}$ from the beginning.

$a_{83} = 3 + (83 + 1)4 = 331$.

Selection of terms in an A.P.

Sometimes we require certain number of terms in A.P. The following ways of selecting terms are generally very convenient.

No. of Terms	Terms	Common Difference
For 3 terms	a – d, a, a + d	d
For 4 terms	a – 3d, a – d, a + d, a + 3d	2d
For 5 terms	a – 2d, a – d, a, a + d, a + 2d	d
For 6 terms	a – 5d, a – 3d, a – d, a + d, a + 3d, a + 5d	2d

Example:

The sum of three numbers in A.P. is -3 and their product is 8 . Find the numbers.

Solution: Three numbers in A.P. be $a - d, a, a + d$

$\therefore a - d + a + a + d = -3$

$3a = -3 \Rightarrow a = -1$

and, $(a - d)a(a + d) = 8 \Rightarrow a(a^2 - d^2) = 8$

(-1) $(1 - d^2) = 8 \Rightarrow 1 - d^2 = -8$

$\Rightarrow d^2 = 9 \Rightarrow d = \pm 3$

If $a = 8$ and $d = 3$ numbers are $-4, -1, 2$.

If $a = 8$ and $d = -3$ numbers are $2, -1, -4$.

Example:

Find four numbers in AP, whose sum is 20 and the sum of whose squares is 120 .

Solution: Let the numbers be $(a - 3d), (a - d), (a + d), (a + 3d)$. Then, Sum $= 20$

$\Rightarrow (a - 3d) + (a - d) + (a + d) + (a + 3d) = 20 \Rightarrow 4a = 20 \Rightarrow a = 5$

Now sum of the squares $= 120$

$\Rightarrow (a - 3d)^2 + (a - d)^2 + (a + d)^2 + (a + 3d)^2 = 120$

$\Rightarrow 4a^2 + 20d^2 = 120 \Rightarrow a^2 + 5d^2 = 30 \Rightarrow 25 + 5d^2 = 30$

$\Rightarrow 25 + 5d^2 = 30 \Rightarrow 5d^2 = 5 \Rightarrow d = \pm 1$

If $d = 1$, then the numbers are $2, 4, 6, 8$. If $d = -1$, then the numbers are $8, 6, 4, 2$. Thus, the numbers are $2, 4, 6, 8$ or $8, 6, 4, 2$.

Sum of n Terms of an A.P.

Let A.P. be $a, a + d, a + 1d, a + 3d, \ldots\ldots a + (n - 1)d$

Then, $S_n = a + (a + d) + \cdots\ldots + \{a + (n - 2)d\} + \{a + (n - 1)d\}$ ...(1)

also, $S_n = \{a + (n - 1)d\} + \{a + (n - 2)d\} + \cdots\ldots + (a + d) + a$ ...(2)

Add (1) & (2)

$\Rightarrow 2S_n = 2a + (n - 1)d + 2a + (n - 1)d + \cdots\ldots\ldots\ldots + 2a + (n - 1)d$

$\Rightarrow 2S_n = n[2a + (n - 1)d]$

$\Rightarrow S_n = \frac{n}{2}[2a + (n - 1)d]$

$S_n = \frac{n}{2}[a + a + (n - 1)d] = \frac{n}{2}[a + \ell]$

$\therefore S_n = \frac{n}{2}[a + \ell]$ where ℓ is the last term.

Example:

Find the sum of 20 terms of the A.P. 1, 4, 7, 10

Solution:. $a = 1, d = 3$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$S_{20} = \frac{20}{2}[2(1) + (20-1)3] = 590$$

Example:

Find the sum of first 30 terms of an A.P. whose second term is 2 and seventh term is 22 .

Solution: Let a be the first term and d be the common difference of the given A.P. Then,

$a_2 = 2$ and $a_7 = 22 \Rightarrow a + d = 2$ and $a + 6d = 22$

Solving these two equations, we get

$a = -2$ and $d = 4$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$\therefore S_{30} = \frac{30}{2}[2 \times (-2) + (30-1) \times 4]$$

$$\Rightarrow 15(-4 + 116) = 15 \times 112 = 1680$$

Hence, the sum of first 30 terms is 1680 .

Example:

Find the sum of all natural numbers between 250 and 1000 which are exactly divisible by 3 .

Solution: Clearly, the numbers between 250 and 1000 which are divisible by 3 are 252,255 , 258, ...,999.

This is an A.P. with first term

$a = 252$, common difference $= 3$ and last term $= 999$. Let there be n terms in this A.P.

Then,

$$\Rightarrow a_n = 999$$

$$\Rightarrow a + (n-1)d = 999$$

$$\Rightarrow 252 + (n-1) \times 3 = 999 \Rightarrow n = 250$$

$\therefore$ Required sum $= S_n = \frac{n}{2}[a + l]$

$$= \frac{250}{2}[252 + 999] = 156375$$

Example:

How many terms of the series 54, 51, 48, ... be taken so that their sum is 513 ? Explain the double answers.

Solution: $\because a = 54, d = -3$ and $S_n = 513$

$$\Rightarrow \frac{n}{2}[2a + (n-1)d] = 513$$

$$\Rightarrow \frac{n}{2}[108 + (n-1) \times -3] = 513$$

$$\Rightarrow n^2 - 37n + 342 = 0$$

$$\Rightarrow (n-18)(n-19) = 0 \Rightarrow n = 18 \text{ or } 19$$

Here, the common difference is negative, So, 19^{th} term is $a_{19} = 54 + (19-1) \times -3 = 0$.

Thus, the sum of 18 terms as well as that of 19 terms is 513 .

Example:

If the m^{th} term of an A.P. is $\frac{1}{n}$ and the n^{th} term is $\frac{1}{m}$, show that the sum of mn terms is $\frac{1}{2}(mn + 1)$.

Solution: Let a be the first term and d be the common difference of the given A.P.

Then, $a_m = \frac{1}{n} \Rightarrow a + (m-1)d = \frac{1}{n}$ and $a_n = \frac{1}{m}$

$\frac{1}{m} \Rightarrow a + (n-1)d = \frac{1}{m}$ ... (ii)

Subtracting equation (ii) from equation (i),

we get

$$(m-n)d = \frac{1}{n} - \frac{1}{m}$$

$$\Rightarrow (m-n)d = \frac{m-n}{mn} \Rightarrow d = \frac{1}{mn}$$

Putting $d = \frac{1}{mn}$ in equation (i), we get

$$a + (m-1)\frac{1}{mn} = \frac{1}{n}$$

$$\Rightarrow a + \frac{1}{n} - \frac{1}{mn} = \frac{1}{n} \Rightarrow a = \frac{1}{mn}$$

Now, $S_{mn} = \frac{mn}{2}\{2a + (mn-1) \times d\}$

$\Rightarrow S_{mn} = \frac{mn}{2}\left[\frac{2}{mn} + (mn-1) \times \frac{1}{mn}\right]$

$\Rightarrow S_{mn} = \frac{1}{2}(mn+1)$

Example:

If the term of m terms of an A.P. is the same as the sum of its n terms, show that the sum of its $(m+n)$ terms is zero.

Solution: Let a be the first term and d be the common difference of the given A.P. Then,

$S_m = S_n$

$\Rightarrow \frac{m}{2}[2a + (m-1)d] = \frac{n}{2}[2a + (n-1)d]$

$\Rightarrow 2a(m-n) + \{m(m-1) - n(n-1)\}d = 0$

$\Rightarrow 2a(m-n) + \{(m^2 - n^2) - (m-n)\}d = 0$

$\Rightarrow (m-n)[2a + (m+n-1)d] = 0$

$\Rightarrow 2a + (m+n-1)d = 0$

$\Rightarrow 2a + (m+n-1)d = 0[\because m-n \neq 0] \ldots.(i)$

Now, $S_{m+n} = \frac{m+n}{2}\{2a + (m+n-1)d\}$

$S_{m+n} = \frac{m+n}{n} \times 0 = 0$[Using equation (i)]

Example:

The ratio of the sum use of n terms of two A.P.'s is $(7n+1):(4n+27)$. Find the ratio of their m^{th} terms.

Solution: Let a_1, a_2 be the first terms and d_1, d_2 the common differences of the two given A.P.'s .

Then the sums of their n terms are given by

$S_n = \frac{n}{2}[2a_1 + (n-1)d_1]$, and

$S_n' = \frac{n}{2}[2a_2 + (n-1)d_2]$

$\therefore \frac{S_n}{S_n} = \frac{\frac{n}{2}[2a_1 + (n-1)d_1]}{\frac{n}{2}[2a_2 + (n-1)d_2]} = \frac{2a_1 + (n-1)d_1}{2a_2 + (n-1)d_2}$

It is given that $\frac{S_n}{S_n'} = \frac{7n+1}{4n+27}$

$\Rightarrow \frac{2a_1 + (n-1)d_1}{2a_2 + (n-1)d_2} = \frac{7n+1}{4n+27} \quad \ldots(i)$

To find the ratio of the m^{th} terms of the two given A.P.'s, we replace n by $(2m-1)$ in equation (i). Then we get

To find the ratio of the m^{th} terms of the two given A.P.'s, we replace n by $(2m-1)$ in equation (i). Then we get

$\therefore \frac{2a_1 + (2m-2)d_1}{2a_2 + (2m-2)d_2} = \frac{7(2m-1)+1}{4(2m-1)+27}$

$\Rightarrow \frac{a_1 + (m-1)d_1}{a_2 + (m-1)d_2} = \frac{14m-6}{8m+23}$

Hence the ratio of the m^{th} terms of the two A.P.'s is $(14m-6):(8m+23)$

(1) The n^{th} term of the AP $a, 3a, 5a, \ldots$ is

(A) na (B) $(2n-1)a$

(C) $(2n+1)a$ (D) $2na$

(2) The common difference of the AP $\frac{1}{p}, \frac{1-p}{p}, \frac{1-2p}{p}, \ldots$ is

(A) 1 (B) $\frac{1}{p}$

(C) -1 (D) $-\frac{1}{p}$

(3) The value of x for which $2x, (x+10)$ and $(3x+2)$ are the three consecutive terms of an AP, is
(A) 6 (B) -6
(C) 18 (D) -18

(4) The first term of AP is p and the common difference is q, then its 10th term is
(A) $q+9p$ (B) $p-9q$
(C) $p+9q$ (D) $2p+9q$

(5) In an AP, if $d=-4, n=7$ and $a_n=4$, then a is equal to
(A) 6 (B) 7
(C) 20 (D) 28

(6) In an AP, if $a=3.5, d=0$ and $n=101$, then a_n will be
(A) 0 (B) 3.5
(C) 103.5 (D) 104.5

(7) The 11th term of an AP $-5, \frac{-5}{2}, 0, \frac{5}{2}, \ldots$, is
(A) -20 (B) 20
(C) -30 (D) 30

(8) If the first term of an AP is -5 and the common difference is 2, then the sum of the first 6 terms is
(A) 0 (B) 5
(C) 6 (D) 15

(9) The sum of first 16 terms of the AP $10,6,2, \ldots \ldots$ is
(A) -320 (B) 320
(C) -352 (D) -40

(10) In an AP, if $a=1, a_n=20$ and $S_n=399$, then n is equal to
(A) 19 (B) 21
(C) 38 (D) 42

Answer Key

(1) (B)	**(2)** (C)	**(3)** (A)	**(4)** (C)	**(5)** (D)
(6) (B)	**(7)** (B)	**(8)** (A)	**(9)** (A)	**(10)** (C)

Properties of A.P.

(A) For any real numbers a and b, the sequence whose n^{th} term is $a_n = an + b$ is always an A.P. with common difference ' a ' (i.e. coefficient of term containing n)
(B) If any n^{th} term of sequence is a linear expression in n then the given sequence is an A.P.
(C) If a constant term is added to or subtracted from each term of an A.P. then the resulting sequence is also an A.P. With the same common difference.
(D) If each term of a given A.P. is multiplied or divided by a non-zero constant K, then the resulting sequence is also an A.P. with common difference Kd or $\frac{d}{K}$
respectively. Where d is the common difference of the given A.P.
(E) In a finite A.P. the sum of the terms equidistant from the beginning and end is always same and is equal to the sum of first and last term.
(F) If three numbers a, b, c are in A.P., then $2b = a + c$.

Example:

200 logs are stacked in the following manner: 20 logs in the bottom row, 19 in the next row, 18 in the row next to it and so on (see figure). In how many rows are the 200 logs placed and how many logs are in the top row?

Solution: The number of logs in the bottom row, next row, row next to it and so on form the sequence
$20, 19, 18, 17, \ldots \ldots$
$a_2 - a_1 = 19 - 20 = -1$
$a_3 - a_2 = 18 - 19 = -1$
$a_4 - a_3 = 17 - 18 = -1$
i.e., $a_k + 1 - a_k$ is the same everytime.
So, the above sequence forms an AP.
Here, $a = 20$
$d = -1$
$S_n = 200$
We know that
$S_n = \frac{n}{2}[2a + (n-1)d]$
$\Rightarrow 200 = \frac{n}{2}[2(20) + (n-1)(-1)] \Rightarrow 200 = \frac{n}{2}[40 - n + 1]$
$\Rightarrow 200 = \frac{n}{2}[41 - n] \Rightarrow 400 = n[41 - n] \Rightarrow n[41 - n] = 400 \Rightarrow 41n - n^2 = 400$

$\Rightarrow n^2 - 41n + 400 = 0 \Rightarrow n^2 - 25n - 16n + 400 = 0$
$\Rightarrow n(n - 25n) - 16(n - 25) = 0 \Rightarrow (n - 25)(n - 16) = 0$
$\Rightarrow n - 25 = 0$ or $n - 16 = 0 \Rightarrow n = 25$ or $n = 16 \Rightarrow n = 25,16$
Hence, the number of rows is either 25 or 16.
Now, number of logs in row = Number of logs in 25th row $= a_{25}$
$= a + (25 - 1)d\ [\because a_n = a + (n - 1)d]$
$= a + 24d$
$= 20 + 24(-1) \Rightarrow 20 - 24 = -4$
Which is not possible.
Therefore, $n = 16$ and
Number of log in top row = Number of logs in 16th row $= a_{16}$
$= a + (16 - 1)d[\because a_n = a + (n - 1)d]$
$= a + 15d$
$= 20 + 15(-1) = 20 - 15 = 5$
Hence, the 200 logs are placed in 16 rows and there are 5 logs in the top row.

Solved Exercise

Direction: In the following questions, a statement of assertion (A) is followed by a statement of reason (R). Mark the correct choice as:
(A) Both assertion (A) and reason (R) are true and reason (R) is the correct explanation of assertion (A).
(B) Both assertion (A) and reason (R) are true but reason (R) is not the correct explanation of assertion (A).
(C) Assertion (A) is true but reason (R) is false.
(D) Assertion (A) is false but reason (R) is true.

(1) Assertion: Let the positive numbers a, b, c be in A.P., then $\frac{1}{bc}, \frac{1}{ac}, \frac{1}{ab}$ are also in A.P.
Reason: If each term of an A.P. is divided by abc, then the resulting sequence is also in A.P.
Solution: (a) Both assertion (A) and reason (R) are true and reason (R) is the correct explanation of assertion (A).

(2) Assertion: Sum of first 10 terms of the arithmetic progression $-0.5, -1.0, -1.5, \ldots\ldots\ldots$ is -27.5
Reason: Sum of n terms of an A.P. is given as $S_n = \frac{n}{2}[2a + (n-1)d]$ where a = first term, d = common difference.
Solution: (a) Both assertion (A) and reason (R) are true and reason (R) is the correct explanation of assertion (A).
Both are correct. Reason is the correct reasoning for Assertion.
Assertion,
$S_{10} = \frac{10}{2}[2(-0.5) + (10-1)(-0.5)]$
$= 5[-1 - 4.5]$,
$= 5(-5.5) = -27.5$

(3) Assertion: The sum of the series with the nth term $t_n = (9 - 5n)$ is (465), when number of terms $n = 15$.
Reason: Given series is in A.P. and sum of n terms of an A.P. is $S_n = \frac{n}{2}[2a + (n-1)d]$
Solution: (d) Assertion (A) is false but reason (R) is true.

(4) Assertion: Three consecutive terms $2k + 1, 3k + 3$ and $5k - 1$ form an AP then k is equal to 6
Reason: In an AP $a, a + d, a + 2d, \ldots\ldots\ldots$, the sum to n terms of the AP be $S_n = \frac{n}{2}(2a + (n-1)d)$
Solution: (b) Both assertion (A) and reason (R) are true but reason (R) is not the correct explanation of assertion (A).
For $2k + 1, 3k + 3$ and $5k - 1$ to form an AP
$(3k + 3) - (2k + 1) = (5k - 1) - (3k + 3)$
$k + 2 = 2k - 4$
$2 + 4 = 2k - k = k$
$k = 6$
So, both A and R are correct but R does not explain A.

(5) Assertion: If S_n is the sum of the first n terms of an A.P., then its n^{th} term a_n is given by $a_n = S_n - S_{n-1}$.
Reason: The 10^{th} term of the A.P. $5, S, 11, 14, \ldots\ldots\ldots$is 35.
Solution: (c) Assertion (A) is true but reason (R) is false.
$a_{10} = a + 9d$
$= 5 + 9(3) = 5 + 27 = 32$

(6) Assertion: Common difference of an AP in which $a_{21} - a_7 = 84$ is 14.
Reason: nth term of AP is given by $a_n = a + (n-1)d$
Solution: (d) Assertion (A) is false but reason (R) is true. Assertion is incorrect.
We have,
$a_n = a + (n-1)d$
$a_{21} - a_7 = \{a + (21-1)d\}$
$-\{a + (7-1)d\} = 84$
have, $a + 20d - a - 6d = 84$
$14d = 84$
$d = \frac{18}{14} = 6$
$d = 6$
So, A is incorrect but R is correct

(7) Find the 9th term from the end (towards the first term) of the A.P. 5, 9,13, ….185.
Solution: Reversing the given A.P., we get
$185, 181, 174, \ldots, 9, 5$
Now, first term $(a) = 185$
Common difference, $(d) = 181 - 185 = -4$
We know that nth term of an A.P. is given by $a + (n-1)d$
Ninth term $a_9 = a + (9-1)d$
$= 185 + 8 \times (-4) = 185 - 32 = 153$

(8) For what value of k will the consecutive terms $2k + 1, 3k + 3$ and $5k - 1$ form an A.P.?
Solution: Given that $2k + 1, 3k + 3$ and $5k - 1$ are in A.P.

So, $(3k+3)-(2k+1) = (5k-1)-(3k+3)$
$\Rightarrow k+2 = 2k-4$
$\Rightarrow 2k-k = 2+4 \Rightarrow k = 6$

(9) Find the 25th term of the A.P. $-5, -\frac{5}{2}, 0, \frac{5}{2}$ …………

Solution: We have,

first term $(a) = -5$, second term $(a_2) = \frac{-5}{2}$, third term $(a_3) = 0$

$d = \frac{-5}{2} - (-5) = \frac{5}{2}$

Now, we know that

$a_n = a + (n-1)d$

$a_{25} = a + 24d = -5 + 24 \times \frac{5}{2} = 55$

(10) Find the sum of all three-digit natural numbers, which are multiples of 9.

Solution: 3-digit natural numbers which are multiples of 9 are $108, 117, \ldots, 999$

It forms an AP with $a = 108, d = 9, a_n = 999$

$\therefore$ n^{th} term, $a_n = a + (n-1)d$

$\Rightarrow 999 = 108 + (n-1) \times 9 \Rightarrow 999 - 108 = (n-1) \times 9$

$\Rightarrow (n-1) = \frac{891}{9} = 99 \Rightarrow n = 99 + 1 = 100$

$S_{100} = \frac{100}{2}(108 + 999) = 55350$

$\therefore$ Sum of all 3-digit natural numbers, multiples of 9 is 55350.

Exercise

OBJECTIVE TYPE QUESTIONS

(1) If the sum of first n terms of an AP be $3n^2 - n$ and it's common difference is 6, then its first term is :
(A) 2 (B) 3
(C) 1 (D) 4

(2) If 7^{th} and 13^{th} terms of an A.P. be 34 and 64 , respectively, then it's 18^{th} term is :
(A) 87 (B) 88
(C) 89 (D) 90

(3) The sum of all 2-digit odd numbers is :
(A) 2475 (B) 2530
(C) 4905 (D) 5049

(4) The fourth term of an A.P. is 4. Then the sum of the first 7 terms is :
(A) 4 (B) 28
(C) 16 (D) 40

(5) In an A.P. $s_1 = 6, s_7 = 105$, then $s_n : s_{n-3}$ is same as :
(A) $(n+3):(n-3)$ (B) $(n+3):n$
(C) $n:(n-3)$ (D) None of these

(6) In an A.P. $s_3 = 6, S_6 = 3$, then it's common difference is equal to :
(A) 3 (B) -1
(C) 1 (D) None of these

(7) The number of terms common to the two A.P.s $2+5+8+11+\cdots+98$ and $3+8+13+18+\cdots+198$
(A) 33 (B) 40
(C) 7 (D) None of these

(8) $(p+q)$ th and $(p-q)$ th terms of an A.P. are respectively m and n, The P^{th} term is :
(A) $\frac{1}{2}(m+n)$ (B) $\sqrt{mn}$
(C) $m+n$ (D) mn

(9) The first, second and last terms of an A.P. are a, b and $2a$. The number of terms in the A.P. is:
(A) $\frac{b}{b-a}$ (B) $\frac{b}{b+a}$
(C) $\frac{a}{b-a}$ (D) $\frac{a}{a+b}$

(10) Let s_1, s_2, s_3 be the sums of n terms of three series in A.P., the first term of each being 1 and the common differences $1,2,3$ respectively. If $s_1 + s_3 = \lambda s_2$, then the value of λ is :
(A) 1 (B) 2
(C) 3 (D) None of these

(11) Sum of first 5 terms of an A.P. is one fourth of the sum of next five terms. If the first term $= 2$, then the common difference of the A.P. is :
(A) 6 (B) -6
(C) 3 (D) None of these

(12) If x, y, z are in A.P., then the value of $(x+y-z)(y+z-x)$ is equal to:
(A) $8yz - 3y^2 - 4z^2$ (B) $8yz - 3z^2 - 4y^2$
(C) $8yz + 3y^2 - 4z^2$ (D) $8yz - 3y^2 + 4z^2$

(13) The number of numbers between 105 and 1000 which are divisible by 7 is :
(A) 142 (B) 128
(C) 127 (D) None of these

(14) If the numbers a, b, c, d, e form an A.P. then the value of $a - 4b + 6c - 4d + e$ is equal to:
(A) 1 (B) 2
(C) 0 (D) None of these

(15) If s_n denotes the sum of first n terms of an A.P., whose common difference is d, then $s_n - 2s_{n-1} + s_{n-2} (n > 2)$ is equal to :
(A) $2d$ (B) $-d$
(C) d (D) None of these

(16) The sum of all 2-digit numbers which leave remainder 1 when divided by 3 is:
(A) 1616 (B) 1602
(C) 1605 (D) None of these

(17) The first term of an A.P. of consecutive integers is $p^2 + 1$. The sum of $2p + 1$ terms of this series can be expressed as :

(A) $(p+1)^2$ (B) $(2p+1)(p+1)^2$
(C) $(p+1)^3$ (D) $p3+(p+1)^3$

(18) If the sum of n terms of an AP is $2n^2+5n$, then its nth term is –
(A) $4n-3$ (B) $3n-4$
(C) $4n+3$ (D) $3n+4$

(19) If the last term of an AP is 119 and the 8th term from the end is 91 then the common difference of the AP is–
(A) 2 (B) 4
(C) 3 (D) -3

(20) If $(a_n)=\{2.5, 2.51, 2.52, \ldots\}$ and $\{b_n\}=\{3.72, 3.73, 3.74, \ldots\}$ be two AP's then $a_{100005}-b_{100005}=$
(A) -1.22 (B) 1.22
(C) 1.2 (D) -1.02

Answer Key

OBJECTIVE TYPE QUESTIONS

(1)	**(2)**	**(3)**	**(4)**	**(5)**	**(6)**	**(7)**	**(8)**	**(9)**	**(10)**
(A)	(C)	(A)	(B)	(A)	(B)	(C)	(A)	(A)	(B)
(11)	**(12)**	**(13)**	**(14)**	**(15)**	**(16)**	**(17)**	**(18)**	**(19)**	**(20)**
(B)	(A)	(C)	(C)	(C)	(C)	(D)	(C)	(B)	(A)