INTRODUCTION TO TRIGONOMETRY

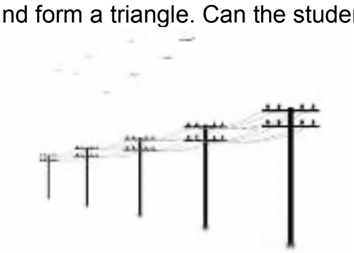
Concepts Covered

- Introduction to Trigonometry and study the relationship between side and angle of a triangle. Distinguish various trigonometric ratios and describe and verify sine, cosine, tangent, cosecant, secant, cotangent of an angle. Use given trigonometric ratio(s) and find and verify other trigonometric ratios /angles of the triangle.

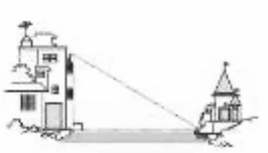
Introduction

We have learnt about triangles and their properties in previous classes. There, we observed different daily life situations where triangles are used. Let's look at some of the daily life examples:

- Electric poles are present everywhere. They are usually erected by using a metal wire. The pole, wire and the ground form a triangle. Can the student find out the height of the Pole, without actually measuring it?



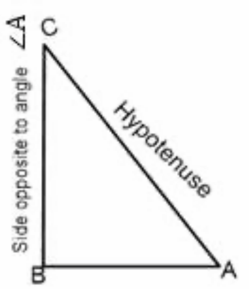
- Suppose a girl is sitting on the balcony of her house located on the bank of a river. She is looking down at a flowerpot placed on a stair of a temple situated nearby on the other bank of the river. A right triangle is imagined to be made in this situation as shown.



- In a playground, children like to slide on a slider and slider is on a defined angle from earth. What will happen to the slider if we change the angle? Will children still be able to play on it?

In all the situation given above, the distance or heights or slopes can be found by using some mathematical techniques, which come under a branch of Mathematics called ''Trigonometry". The Word 'Trigonometry' is derived from the Greek Words 'Tri' (meaning three), 'gon' (meaning sides) and 'metron' (meaning measure). In fact, trigonometry is the study of relationships between the sides and angles of a triangle. Early astronomers used it to find out the distances of the stars and planets from the Earth. Even today, most of the technologically advanced methods used in Engineering and Physical Sciences are based on trigonometrical concepts.

In this chapter, we will study some ratios of the sides of a right angled triangle with respect to its acute angles, called "Trigonometric ratios of the angle".



Trigonometric Ratio

Let's take a right triangle ΔABC which is right angled at vertex B

The Trigonometric ratio of the angle A in right angle triangle ABC which is right angled at B .

Sine of $\angle A = \frac{\text{side Opposite to } \angle A}{\text{Hypotenuse}} = \frac{BC}{AC}$

Cosine of $\angle A = \frac{\text{side adjacent to } \angle A}{\text{Hypertenuse}} = \frac{BA}{AC}$

Tangent of $\angle A = \frac{\text{side opposite to } \angle A}{\text{Side adjacent to } \angle A} = \frac{BC}{BA}$

Cotangent of $\angle A = \frac{1}{\text{Tan of } \angle A} = \frac{\text{Side adjacent to } \angle A}{\text{Side opposite to } \angle A} = \frac{BA}{BC}$

Secant of $\angle A = \frac{1}{\text{Cosine of } \angle A} = \frac{\text{Hypotenuse}}{\text{side adjacent to } \angle A} = \frac{AC}{BA}$

Cosecant of $\angle A = \frac{1}{\text{Sine of } \angle A} = \frac{\text{Hypotenuse}}{\text{side Opposite to } \angle A} = \frac{AC}{BC}$

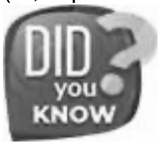
The ratios defined above are abbreviated as $\sin A, \cos A, \tan A, \cot A, \sec A$ and $\operatorname{cosec} A$ respectively. Note that the ratios cosec A, sec A and cot A are respectively, the reciprocals of the ratios sin A, cos A and tan A..

Also, observe that $\tan A = \frac{BC}{AB} = \frac{\frac{BC}{AC}}{\frac{AB}{AC}} = \frac{\sin A}{\cos A}$ and $\cot A = \frac{\cos A}{\sin A}$.

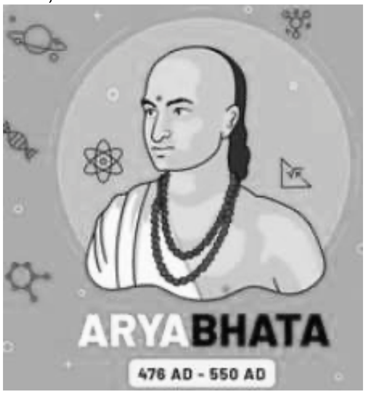
So, the trigonometric ratios of an acute angle in a right triangle express the relationship between the angle and the length of its sides.

Note:- The symbol sin A is used as an abbreviation for the 'sine of angle $\angle A$ '. Sin A is not the product of 'sin' and 'A'. 'Sin' separated from 'A' has no meaning. Similarly, cos A is not the product of 'cos' and A. Similar interpretation follow for other trigonometric ratios also.

Note : Since the hypotenuse is the longest side in a right triangle, the value of sin A or cos A is always less than 1 (or, in particular, equal to 1).



The first use of the idea of 'sine' in the way we use it today was in the work Aryabhatiyam by Aryabhata, in A.D. 500. Aryabhata used the word ardha-jya for the half-chord, which was shortened to jya or jiva in due course. When the Aryabhatiyam was translated into Arabic, the word jiva was retained as it is. The word jiva was translated into sinus, which means curve, when the Arabic version was translated into Latin. Soon the word sinus, also used as sine, became common in mathematical texts throughout Europe.



Example:

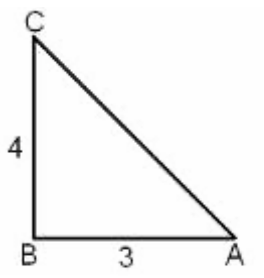
In $\triangle ABC$, right angled at $B, AB = 3, BC = 4$ then determine all trigonometric ratio of angle A.

Solution: Now by using Pythagoras Theorem

$AC^2 = AB^2 + BC^2$

$AC^2 = 3^2 + 4^2$

AC = 5



Now, $\sin A = \frac{\text{Side opposite of angle } A}{\text{Hypotenuse}}$

$\sin A = \frac{BC}{AC} = \frac{4}{5}$

Now $\cos A = \frac{AB}{AC} = \frac{3}{5}$

Now,

$\tan A = \frac{BC}{AB} = \frac{4}{3}$

$\cot A = \frac{AB}{BC} = \frac{3}{4}$

$$\sec A = \frac{AC}{AB} = \frac{5}{3}$$
$$\operatorname{cosec} A = \frac{AC}{BC} = \frac{5}{4}$$

Example:

If θ is acute angle for $\tan\theta = \frac{5}{12}$ then find the other trigonometric ratios of the angle θ.

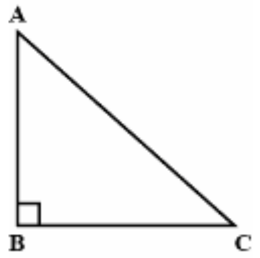
Solution: Let us first draw a right $\triangle ABC$,right angled at $\angle B$ and . let one acute angle ,$\angle C$ be θ

$$\tan\theta = \frac{AB}{BC} = \frac{5}{12}$$

Therefore if $AB = 5k$ then $BC = 12k$ where k is positive number

Now by using the Pythagoras Theorem, we have

$$AC^2 = AB^2 + BC^2$$
$$= (5k)^2 + (12k)^2 = 169k^2 \text{ so, } AC = 13k$$



Now we can write all the trigonometric ratio using their definitions.

$$\sin\theta = \frac{AB}{AC} = \frac{5k}{13k} = \frac{5}{13}$$
$$\cos\theta = \frac{BC}{AC} = \frac{12k}{13k} = \frac{12}{13}$$
$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{\frac{5}{12}} = \frac{12}{5}$$
$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{12}{13}} = \frac{13}{12}$$
$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{5}{13}} = \frac{13}{5}$$

Example:

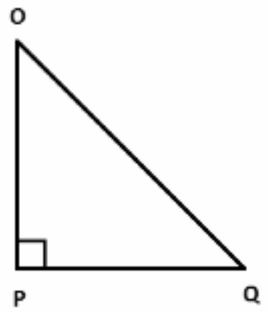
In $\triangle OPQ$, right-angled at 'P', $OP = 7$ cm and $OQ - PQ = 1$ cm then determine the values of $\sin\theta$ and $\tan\theta$.

Solution: In a right angled $\triangle OPQ$, right angled at ' P ' we have

$$OQ^2 = OP^2 + PQ^2$$
$$\Rightarrow (1 + PQ)^2 = OP^2 + PQ^2$$
$$\Rightarrow 1 + PQ^2 + 2PQ = OP^2 + PQ^2$$
$$\Rightarrow 1 + 2PQ = OP^2$$
$$\Rightarrow 1 + 2PQ = 7^2$$
$$\Rightarrow 1 + 2PQ = 49$$
$$\Rightarrow PQ = \frac{48}{2}$$

$PQ = 24$ and $OQ = 1 + PQ = 25$

$$\text{Now, } \sin O = \frac{\text{side of opposite to angle O}}{\text{Hypotenuse}} = \frac{PQ}{OQ} = \frac{24}{25}$$
$$\tan Q = \frac{\text{side of opposite to angle Q}}{\text{side adjacent to angle Q}} = \frac{OP}{PQ} = \frac{7}{24}$$



Example:

If $\angle B$ and $\angle Q$ are acute angles such that $\sin B = \sin Q$. then prove that $\angle B = \angle Q$.

Solution : Let us consider two right triangles ABC and PQR where $\sin B = \sin Q$(sec Fig.).

We have

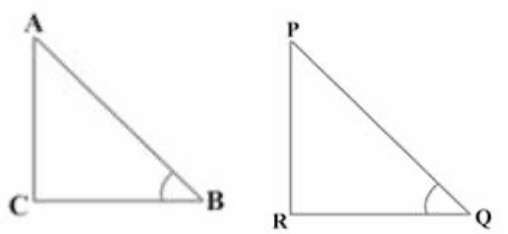
and $\sin B = \frac{AC}{AB}$

$$\sin Q = \frac{PR}{PQ}$$

Then $\frac{AC}{AB} = \frac{PR}{PQ}$

Therefore,

$$\frac{AC}{PR} = \frac{AB}{PQ} = k, \text{ say}$$



Now, using Pythagoras theorem,

$$BC = \sqrt{AB^2 - AC^2}$$
$$QR = \sqrt{PQ^2 - PR^2}$$

So, $\frac{BC}{QR} = \frac{\sqrt{AB^2 - AC^2}}{\sqrt{PQ^2 - PR^2}} = \frac{\sqrt{k^2PQ^2 - k^2PR^2}}{\sqrt{PQ^2 - PR^2}} = \frac{k\sqrt{PQ^2 - PR^2}}{\sqrt{PQ^2 - PR^2}} = k$

From (1) and (2), we have

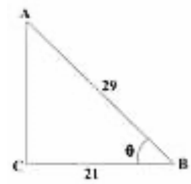
$\frac{AC}{PR} = \frac{AB}{PQ} = \frac{BC}{QR}$

by using theorem which states that " if in two triangles , sides of one triangle are proportional to (i.e., in the same ratio of) the sides of other triangle, then their corresponding angles are equal and hence the two triangles are similar.

$\triangle ACB \sim \triangle PRQ$ and therefore, $\angle B = \angle Q$

Example:

Consider $\triangle ACB$, right-angled at C, in which $AB = 29$ units, $BC = 21$ units and $\angle ABC = \theta$.
Determine the values of
(i) $\cos^2\theta + \sin^2\theta$,
(ii) $\cos^2\theta - \sin^2\theta$.



Solution: In Δ ACB, we have

$AC = \sqrt{AB^2 - BC^2} = \sqrt{(29)^2 - (21)^2}$

$= \sqrt{(29-21)(29+21)} = \sqrt{(8)(50)} = \sqrt{400} = 20$ units

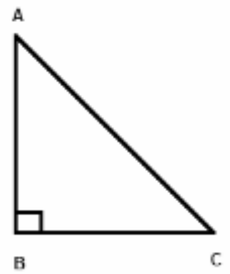
So, $\sin\theta = \frac{AC}{AB} = \frac{20}{29}$, $\cos\theta = \frac{BC}{AB} = \frac{21}{29}$.

Now, (i) $\cos^2\theta + \sin^2\theta = \left(\frac{20}{29}\right)^2 + \left(\frac{21}{29}\right)^2 = \frac{20^2+21^2}{29^2} = \frac{400+441}{841} = 1$,

and (ii) $\cos^2\theta - \sin^2\theta = \left(\frac{21}{29}\right)^2 - \left(\frac{20}{29}\right)^2 = \frac{(21+20)(21-20)}{29^2} = \frac{41}{841}$.

Trigonometric Ratio of Some Specific Angle

Trigonometric Ratios of 45°



In $\triangle ABC$, right angled at B, if one angle is 45°, then the other angle is also 45°

So, $AB = BC = a$ (say)

{AB = BC as when two angles of a triangle are equal, then opposite sides to them are also equal}

Then by Pythagoras Theorem,

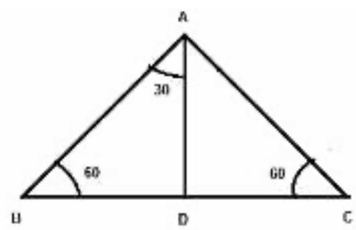
$AC^2 = AB^2 + BC^2 \Rightarrow AC = a^2 + a^2 = 2a^2 \Rightarrow AC = a\sqrt{2}$

Using the definitions of Trigonometric Ratio, we have (with respect to $\angle C$)

$\sin 45^\circ = \frac{AB}{AC} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$, $\cos 45^\circ = \frac{BC}{AC} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$, $\tan 45^\circ = \frac{AB}{BC} = \frac{a}{a} = 1$

Also, $\text{cosec}\, 45^\circ = \frac{1}{\sin 45} = \sqrt{2}$, $\sec 45^\circ = \frac{1}{\cos 45} = \sqrt{2}$ and $\cot 45^\circ = \frac{1}{\tan 45} = 1$

Trigonometric Ratios of 60° and 30°



Consider an equilateral triangle $\triangle ABC$ therefore $\angle A = \angle B = \angle C = 60^\circ$

Draw the perpendicular AD from A to the side BC

Now $\Delta ABD \cong \triangle ACD$ (by RHS congruence rule)

Therefore, $BD = DC$

and $\angle BAD = \angle CAD (CPCT)$

Now observe that:

$\triangle ABD$ is a right triangle, right- angled at D with $\angle BAD = 30^\circ$ and $\angle ABD = 60^\circ$

As we know, for finding the trigonometric ratios, we need to know the lengths of the sides of the triangle

So, Let us suppose $AB = 2a$

$BD = \frac{BC}{2} = \frac{AB}{2} = a$

By using Pythagoras Theorem

$AD^2 = AB^2 - BD^2 \Rightarrow AD^2 = 4a^2 - a^2 = 3a^2 \Rightarrow AD = \sqrt{3}a$

Using the definitions of Trigonometric Ratios we have

$\sin 30^\circ = \frac{BD}{AB} = \frac{a}{2a} = \frac{1}{2}$, $\cos 30^\circ = \frac{AD}{AB} = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2}$, $\tan 30^\circ = \frac{BD}{AD} = \frac{a}{\sqrt{3}a} = \frac{1}{\sqrt{3}}$

Also, $\text{cosec}\, 30^\circ = \frac{1}{\sin 30} = 2$, $\sec 30^\circ = \frac{1}{\cos 30} = \frac{2}{\sqrt{3}}$ and $\cot 30^\circ = \frac{1}{\tan 30} = \sqrt{3}$

Similarly, $\sin 60^\circ = \frac{AD}{AB} = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2}$

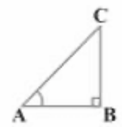
Also, $\cos 60^\circ = \frac{BD}{AB} = \frac{1}{2}$, $\tan 60^\circ = \frac{AD}{BD} = \sqrt{3}$,

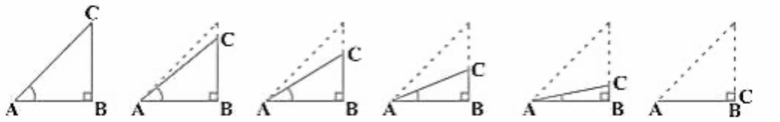
$\operatorname{cosec} 60° = \frac{2}{\sqrt{3}}, \sec 60° = 2$ and $\cot 60° = \frac{1}{\sqrt{3}}$. (By using reciprocals of ratio sin60°,cos60° and tan60 respectively)

Trigonometric Ratios of $0°$ and $90°$

For 0°

Consider a right angle triangle ABC, if ∠A is made smaller and smaller till it becomes zero. As ∠A gets smaller and smaller, the length of the side BC decreases. The point C gets closer to point B, and finally when ∠A becomes very close to 0°, AC becomes almost the same as AB (look at the figure below)

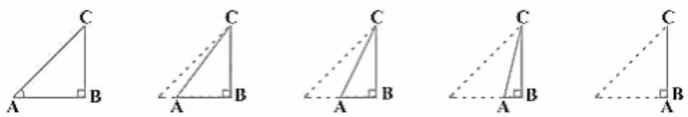




So, we can say When ∠A is very close to 0°, BC gets very close to 0 and
Therefore, sin A = $\frac{BC}{AC}$ is very close to 0 (i)
Also, when ∠A is very close to 0°, AC is almost same as AB and
Therefore, value of cos A = $\frac{AB}{AC}$ is very close to 1 (ii)
So, finally we can define sin 0° and cos 0°
When $\angle A = 0^o,\ \sin 0° = 0$ and $\cos 0^o = 1$ (from (i) and (ii) respectively)
Using these,
we have: tan 0° = $\frac{\sin 0°}{\cos 0°}$ = 0,
cot 0° = $\frac{1}{\tan 0°} = \frac{1}{0}$, which is not defined.
sec 0° $\frac{1}{\cos 0°}$ = = 1 and cosec 0° = $\frac{1}{\sin 0°} = \frac{1}{0}$, which is not defined.

For 90°

Consider a right angle triangle ABC (as shown), if ∠A is made larger and larger till it becomes 90°. As ∠A gets larger and larger, ∠C gets smaller and smaller. Therefore, the length of the side AB goes on decreasing. The point A gets closer to point B. Finally, when ∠A is very close to 90°, ∠C becomes very close to 0° and the side AC almost coincides with side BC (look at the figure below).



When ∠C is very close to 0°, ∠A is very close to 90°, side AC is nearly the same as side BC, and
so sin A = $\frac{BC}{AC}$ is very close to 1. (i)
Also when ∠A is very close to 90°, ∠C is very close to 0°, and the side AB is nearly zero,
so cos A = $\frac{AB}{AC}$ is very close to 0. (ii)
finally, we can define : sin 90° = 1 and cos 90° = 0. (from (i) and (ii) respectively)
Using these,
we have: tan 90° = $\frac{\sin 90°}{\cos 90°} = \frac{1}{0}$, which is not defined
cot 90° = $\frac{1}{\tan 90°} = \frac{1}{\frac{1}{0}}$ = 0 .
sec 90° = $\frac{1}{\cos 90°} = \frac{1}{0}$, which is not defined and cosec 90° = $\frac{1}{\sin 90°} = 1$.
Now, let us see the values of trigonometric ratios of all the above discussed angles in the form of a table.

$\angle\theta$	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined
$\text{cosec}\theta$	Not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Not defined
$\cot\theta$	Not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

Example:

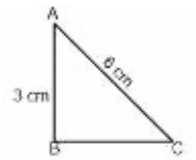
In $\triangle ABC$, right angled at B, $AB = 3$ cm, $AC = 6$ cm Determine $\angle BAC$ and $\angle ACB$

Solution: Given $AB = 3$ cm. $AC = 6$ cm

$$\sin C = \frac{AB}{AC} = \frac{3}{6} = \frac{1}{2}$$

So, $\angle C = 30^\circ$

and therefore $\angle A = 60^\circ$ (by using angle sum property of Δ)



Example:

In $\triangle ABC$, right angled at B, $AB = 5$ cm, and $\angle ACB = 30^\circ$. Determine the lengths of the sides BC and AC.

Solution: We know that $\sin 30^\circ = \frac{AB}{AC}$

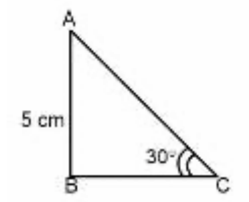
$$\frac{1}{2} = \frac{5}{AC},$$

So, $AC = 10$ cm

Similarly, $\tan 30^\circ = \frac{AB}{BC}$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{5}{BC}$$

Which gives $BC = 5\sqrt{3}$ cm



Example:

If $\sin(A - B) = \frac{1}{2}$, $\cos(A + B) = \frac{1}{2}$, $0^\circ < A + B \le 90^\circ$, $A > B$, find A and B.

Solution : Since, $\sin(A - B) = \frac{1}{2}$,

$\Rightarrow \sin(A - B) = \sin 30^\circ$

$\Rightarrow A - B = 30^\circ$ (1)

Also, since $\cos(A + B) = \frac{1}{2}$

$\Rightarrow \cos(A + B) = \cos 60^\circ$

$\Rightarrow A + B = 60^\circ$ (2)

Solving (1) and (2), we get : $A = 45^\circ$ and $B = 15^\circ$.

Example:

Simplify $\frac{5\cos^2 60^\circ + 4\sec^2 30^\circ - \tan^2 45^\circ}{\sin^2 30^\circ + \cos^2 30^\circ}$

Solution: Putting values $= \frac{5\left(\frac{1}{2}\right)^2 + 4\left(\frac{2}{\sqrt{3}}\right)^2 - (1)^2}{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

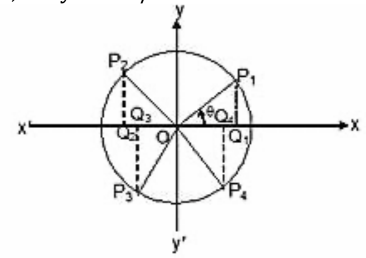
$$= \frac{\frac{5}{4} + 4 \times \frac{2 \times 2}{\sqrt{3} \times \sqrt{3}} - 1}{\frac{1}{4} + \frac{\sqrt{3} \times \sqrt{3}}{2 \times 2}}$$

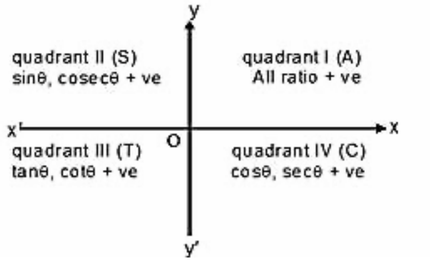
$$= \frac{\frac{5}{4} + \frac{16}{3} - 1}{\frac{1}{4} + \frac{3}{4}}$$

$$= \frac{\frac{15 + 64 - 12}{12}}{\frac{4}{4}} = \frac{67}{12}$$

Trigonometric Ratios of Any Angle

Consider the system of rectangular co-ordinate axes dividing the plane into four quadrants. A line OP_1 makes angle θ with the positive x-axis. The angle θ is said to be positive if measured in counter clockwise direction from the positive x-axis and is negative if measured in clockwise direction. The positive values of the trigonometric ratios in the various quadrants are shown, the signs of the other ratios may be derived. Note that $\angle xoy = \pi/2, \angle xox' = \pi, \angle xoy' = 3\pi/2$





P_iQ_i is positive if above the x-axis, negative if below the x-axis, O is always taken positive. OQ_i is positive if along positive x-axis, negative if in opposite direction.

$\sin\angle Q_iOP_i = \frac{P_iQ_i}{OP_i}, \cos\angle Q_iOP_i = \frac{OQ_i}{OP_i}$ and $\tan\angle Q_iOP_i = \frac{P_iQ_i}{OQ_i}$ (Where $i = 1,2,3,4$).

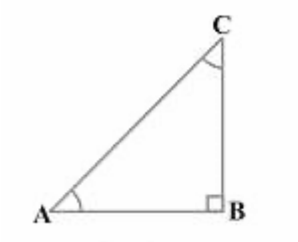
Thus depending on signs of OQ_i and P_iQ_i the various trigonometrical ratios will have different signs.

Table shows signs of trigonometric functions in different Quadrants

	I	II	III	IV
Sin θ	+	+	–	–
Cos θ	+	–	–	+
Tan θ	+	–	+	–
Cosec θ	+	+	–	–
Sec θ	+	–	–	+
Cot θ	+	–	+	–

Complementary angles: Two angles are said to be complementary to each other if their sum is 90°. Hence θ and $(90° - \theta)$ are complementary angles.

In Δ ABC, right-angled at B, we can clearly see pair of complementary angles i.e., ∠A and ∠C (since ,∠A + ∠C = 90° by angle sum property)



Now , we have

$$\left.\begin{aligned} \sin A = \frac{BC}{AC} \quad \cos A = \frac{AB}{AC} \quad \tan A = \frac{BC}{AB} \\ \operatorname{cosec} A = \frac{AC}{BC} \quad \sec A = \frac{AC}{AB} \quad \cot A = \frac{AB}{BC} \end{aligned}\right\} \quad (1)$$

We can write , ∠ C = 90° – ∠ A.

For simplicity, we shall write 90° – A instead of 90° – ∠ A.

Therefore, Sin c, Cos C, tan C, Cosec C, Sec C and Cot C can be written as

$$\left.\begin{aligned} \sin(90° - A) = \frac{AB}{AC}, \ \cos(90° - A) = \frac{BC}{AC}, \quad \tan(90° - A) = \frac{AB}{BC} \\ \operatorname{cosec}(90° - A) = \frac{AC}{AB}, \sec(90° - A) = \frac{AC}{BC}, \ \cot(90° - A) = \frac{BC}{AB} \end{aligned}\right\} \quad (2)$$

Now, compare the ratios in (1) and (2). Observe that :

$\sin(90° - A) = \frac{AB}{AC} = \cos A$ and $\cos(90° - A) = \frac{BC}{AC} = \sin A$

Also, $\tan(90° - A) = \frac{AB}{BC} = \cot A, \ \cot(90° - A) = \frac{BC}{AB} = \tan A$

Table below shows signs of trigonometric function in different quadrants based on complementary angle.

α equals	sin α	cos α	tan α	cot α	sec α	cosec α
–θ	– sinθ	cosθ	–tanθ	– cotθ	secθ	–cosec θ
90° – θ	cosθ	sinθ	cotθ	tanθ	cosecθ	secθ
90° + θ	cosθ	– sinθ	–cotθ	– tanθ	–cosecθ	secθ

Example:

Evaluate $\frac{\tan 65°}{\cot 25°}$

Solution: We know $\cot\theta = (90 - \theta)$,

$\cot 25° = \tan(90 - 25°)$

$= \tan 65°$

$\text{Therefore}, \frac{\tan 65°}{\cot 25°} = \frac{\tan 65°}{\tan 65°} = 1$

Example:

If $\sin 3A = \cos(A - 26)$, where $3A$ is acute angle ,find angle A

Solution: We know $\sin 3A = \cos(90° - 3A)$,

i.e., $\cos(A - 26°) = \cos(90° - 3A)$

i.e.,$A - 26° = 90 - 3A$ (Since both acute angle)

$\Rightarrow A = 29°$

Example:

Express $\sin 67° + \cot 75°$ in terms of trigonometric ratio of angles between $0°$ and $45°$

Solution: $\sin 67° + \cot 75° = \sin(90° - 23°) + \cot(90° - 15°)$

$= \cos 23° + \tan 15°$

Trigonometry Identity

An equation involving trigonometric ratios of an angle is called a trigonometric identity, if it is true for all values of the angle(s) involved.

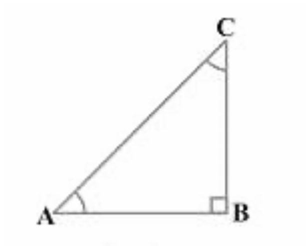
Here, we will be proving one trigonometric identity which will be used further to prove other useful trigonometric identities.

(i) $\text{Sin}^2\theta + \text{Cos}^2\theta = 1$

(ii) $\text{Sec}^2\theta = 1 + \tan^2\theta$

(iii) $\text{Cosec}^2\theta = 1 + \text{Cot}^2\theta$

Consider ΔABC, right angled at B .Here ,



$AB^2 + BC^2 = AC^2$ (1)

Dividing each term of (1) by AC^2, we get

$$\frac{AB^2}{AC^2} + \frac{BC^2}{AC^2} = \frac{AC^2}{AC^2}$$

i.e., $\left(\frac{AB}{AC}\right)^2 + \left(\frac{BC}{AC}\right)^2 = \left(\frac{AC}{AC}\right)^2$

$(\cos A)^2 + (\sin A)^2 = 1$

$\cos^2 A + \sin^2 A = 1$ (2)

This is true for all A such that $0° \leq A \leq 90°$. So, this is a trigonometric identity. Let us now divide (1) by AB^2. We get

$$\frac{AB^2}{AB^2} + \frac{BC^2}{AB^2} = \frac{AC^2}{AB^2}$$

$$\left(\frac{AB}{AB}\right)^2 + \left(\frac{BC}{AB}\right)^2 = \left(\frac{AC}{AB}\right)^2$$

$1 + \tan^2 A = \sec^2 A$ (3)

Since, $\tan A$ and $\sec A$ are not defined for $A = 90°$.

So, equation (3) is true for all A such that $0° \leq A < 90°$.

$$\frac{AB^2}{BC^2} + \frac{BC^2}{BC^2} = \frac{AC^2}{BC^2}$$

$$\left(\frac{AB}{BC}\right)^2 + \left(\frac{BC}{BC}\right)^2 = \left(\frac{AC}{BC}\right)^2$$

i.e., $\cot^2 A + 1 = \text{cosec}^2 A$ (4)

Note that $\text{cosec} A$ and $\cot A$ are not defined for $A = 0°$. Therefore equation (4) is true for all A such that $0° < A \leq 90°$.

Extended Learning

(i) $\text{Sin}2\theta = 2\text{Sin}\theta \text{ Cos}\theta$

(ii) $\text{Cos}2\theta = 2\text{Cos}^2\theta - 1$

(iii) $\text{Cos}2\theta = 1 - 2\text{Sin}^2\theta$

Example:

Prove that $\sec^2\theta + \text{cosec}^2\theta = \sec^2\theta \cdot \text{cosec}^2\theta$

Solution: LHS $= \sec^2\theta + \text{cosec}^2\theta$

$$= \frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}$$

$$= \frac{\sin^2 + \cos^2\theta}{\cos^2\theta \cdot \sin^2\theta}$$

$$= \frac{1}{\cos^2\theta \cdot \sin^2\theta} \text{ (Since, } \sin^2\theta + \cos^2\theta = 1)$$

$= \sec^2\theta \cdot \text{cosec}^2\theta =$ RHS

Example:

Prove that $\sec\theta\,(1-\sin\theta)(\sec\theta+\tan\theta)=1$

Solution: LHS = $\sec\theta\,(1-\sin\theta)(\sec\theta+\tan\theta)$

$= (\sec\theta - \sin\theta.\sec\theta)(\sec\theta+\tan\theta)$

$= \left(\sec\theta - \frac{\sin\theta}{\cos\theta}\right)(\sec\theta+\tan\theta)$

$= (\sec\theta - \tan\theta)(\sec\theta+\tan\theta)$

$= \sec^2\theta - \tan^2\theta = 1$

Therefore LHS = RHS

Example:

Prove that $\frac{\sin\theta-\cos\theta+1}{\sin\theta+\cos\theta-1} = \frac{1}{\sec\theta-\tan\theta}$

Solution: LHS $= \frac{\sin\theta-\cos\theta+1}{\sin\theta+\cos\theta-1}$ (divide by $\cos\theta$ in numerator and denominator)

$= \frac{\frac{\sin\theta-\cos\theta+1}{\cos\theta}}{\frac{\sin\theta+\cos\theta-1}{\cos\theta}}$

$= \frac{\tan\theta-1+\sec\theta}{\tan\theta+1-\sec\theta}$

(Multiply by $(\sec\theta-\tan\theta)$ in numerator and denominator)

$= \frac{(\tan\theta+\sec\theta-1)(\sec\theta-\tan\theta)}{(1-(\sec\theta-\tan\theta))(\sec\theta-\tan\theta)}$

$= \frac{\sec^2\theta-\tan^2\theta-\sec\theta+\tan\theta}{(1-(\sec\theta-\tan\theta))(\sec\theta-\tan\theta)}$

$= \frac{(1-\sec\theta+\tan\theta)}{(1-\sec\theta+\tan\theta)(\sec\theta-\tan\theta)}$ (we know, $\mathrm{Sec}^2\theta = 1+\tan^2\theta$)

$= \frac{1}{\sec\theta-\tan\theta}$ Therefore , LHS =RHS

Example:

Prove that $\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \mathrm{cosec}\,\theta+\cot\theta;\ 0\le\theta\le 90^0$.

Solution : LHS $= \sqrt{\frac{1+\cos\theta}{1-\cos\theta}}$ (multiply numerator and denominator by $\sqrt{(1+\cos\theta)}$)

$= \sqrt{\frac{1+\cos\theta}{1-\cos\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta}} = \sqrt{\frac{(1+\cos\theta)^2}{1-\cos^2\theta}}$

$= \sqrt{\frac{(1+\cos\theta)^2}{\sin^2\theta}}$ (Since , we know , $\mathrm{Sin}^2\theta + \mathrm{Cos}^2\theta = 1$)

$= \frac{1+\cos\theta}{\sin\theta}$

$= \frac{1}{\sin\theta}+\frac{\cos\theta}{\sin\theta} = \mathrm{cosec}\theta+\cot\theta =$ R.H.S.

Example:

Find tan 15°.tan 60°.tan75°

Solution: tan 15°.tan 60°.tan75°

= tan(90° – 75°) .$\sqrt{3}$.tan75°

= cot 75°.$\sqrt{3}$.tan75° (we know , tan (90–θ) = cotθ)

$= \frac{1}{\tan 75°}.\sqrt{3}.\tan 75° = \sqrt{3}$

Solved Example

(1) If $\sin\theta + \text{cosec}\,\theta = 2$, then $\sin^2\theta + \text{cosec}^2\theta =$

(A) 1 (B) 4
(C) 2 (D) None of these

Solution: (C) $(\sin^2\theta + \text{cosec}^2\theta) = (\sin\theta + \text{cosec}\,\theta)^2 - 2\sin\theta\,\text{cosec}\,\theta$
$= 2^2 - 2(1) = 2.$

(2) If $\sin\theta + \cos\theta = m$ and $\sec\theta + \text{cosec}\,\theta = n$, then $n(m+1)(m-1)$ equal to

(A) m (B) n
(C) $2\,m$ (D) $2n$

Solution: (C) $n(m+1)(m-1) = n(m^2 - 1)$

$= (\sec\theta + \text{cosec}\,\theta)\cdot 2\sin\theta\cdot\cos\theta$ $[\because m^2 = 1 + 2\sin\theta\cdot\cos\theta]$

$= (\frac{1}{\cos\theta} + \frac{1}{\sin\theta}).\ 2\sin\theta\cdot\cos\theta$

$= \frac{\sin\theta + \cos\theta}{\sin\theta\cdot\cos\theta}\cdot 2\sin\theta\cdot\cos\theta = 2m$

(3) If $\tan A + \cot A = 4$, then $\tan^4 A + \cot^4 A$ is equal to

(A) 110 (B) 191
(C) 80 (D) 194

Solution: (D) $\tan A + \cot A = 4$ (1)

$\Rightarrow \tan^2 A + \cot^2 A + 2\tan A\cot A = 16$ (on taking square of both sides of (1))

$\Rightarrow \tan^2 A + \cot^2 A = 14$ (2)

$\Rightarrow \tan^4 A + \cot^4 A + 2 = 196$ (on taking square of both sides of (2))

$\Rightarrow \tan^4 A + \cot^4 A = 194$

(4) The value of $6(\sin^6 q + \cos^6 q) - 9(\sin^4 q + \cos^4 q) + 4$ equals to

(A) -3 (B) 0
(C) 1 (D) 3

Solution: (C) $= 6(\sin^6\theta + \cos^6\theta) - 9(\sin^4\theta + \cos^4\theta) + 4$

$= 6[(\sin^2\theta + \cos^2\theta)^3 - 3\sin^2\theta\cos^2\theta(\sin^2\theta + \cos^2\theta)] - 9[(\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cdot\cos^2\theta] + 4$

$= 6[1 - 3\sin^2\theta\cos^2\theta] - 9[1 - 2\sin^2\theta\cos^2\theta] + 4 = 6 - 9 + 4 = 1$

(5) $\tan A + \cot(180° + A) + \cot(90° + A) + \cot(360° - A)$ equal to

(A) 0 (B) $2\tan A$
(C) $2\cot A$ (D) $2(\tan A - \cot A)$

Solution: (A) $\tan A + \cot A + (-\tan A) + (-\cot A) = 0.$

(6) If $4\sin q = 3\cos q$ then $\frac{\sec^2 q}{4[1-\tan^2 q]}$ equals to

(A) $\frac{25}{16}$ (B) $\frac{25}{28}$
(C) $\frac{1}{4}$ (D) 1

Solution: (B) Given $4\sin\theta = 3\cos\theta \Rightarrow \tan\theta = \frac{3}{4}$

The given expression is $\frac{\sec^2\theta}{4[1-\tan^2\theta]} = \frac{1+\tan^2\theta}{4(1-\tan^2\theta)}$

$= \frac{1+\frac{9}{16}}{4\left(1-\frac{9}{16}\right)} = \frac{25}{28}$

(7) If $\sin x + \sin^2 x = 1$ then the value of $\cos^2 x + \cos^4 x + \cot^4 x - \cot^2 x$ is equal to

(A) 1 (B) 0
(C) 2 (D) none of these

Solution: (C) Given $\sin x + \sin^2 x = 1$

$\Rightarrow \sin x = \cos^2 x$ ...(1)

$\Rightarrow \sin^2 x = \cos^4 x$ (on taking square of both sides of (1)

Now, $\cos^2 x + \cos^4 x + \cot^4 x - \cot^2 x$

$= \cos^2 x + \sin^2 x + \frac{\cos^4 x}{\sin^4 x} - \cot^2 x$

$= 1 + \frac{\sin^2 x}{\sin^4 x} - \cot^2 x$ (we know , $\sin^2 x + \cos^2 x = 1$)

$= 1 + \text{cosec}^2 x - \cot^2 x = 1 + 1 = 2$ (we know , $\text{cosec}^2 x - \cot^2 x = 1$)

Exercise

OBJECTIVE TYPE QUESTIONS

(1) If A is a acute angle and given that $\sin A = \frac{3}{4}$ then value of $\cos A$ is equal to

(A) $\frac{3}{4}$ (B) $\frac{4}{5}$

(C) $\frac{\sqrt{7}}{4}$ (D) None of these

(2) In $\triangle ABC$, right angled at $B, AB = 24$ cm, $BC = 7$ cm then the value of $\sin C$ is equal to

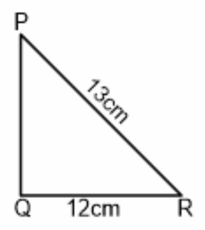
(A) $\frac{7}{25}$ (B) $\frac{7}{24}$

(C) $\frac{24}{25}$ (D) $\frac{12}{25}$

(3) By using figure the value of $\tan P - \tan R$ is equal to

(A) $\frac{119}{60}$ (B) $\frac{60}{156}$

(C) $\frac{156}{60}$ (D) $\frac{60}{119}$



(4) If A is acute angled and given $3\cot A = 4$ then the value of $\frac{1-\tan^2 A}{1+\tan^2 A}$ is equal to

(A) $\frac{7}{25}$ (B) $-\frac{7}{25}$

(C) $\frac{1}{7}$ (D) $-\frac{1}{7}$

(5) In triangle ABC, right-angled at B, if $\tan A = \frac{1}{\sqrt{3}}$ then the value of $\sin A \cdot \cos C + \cos A \sin C$ is equal to

(A) 0 (B) 1

(C) $\sqrt{3}$ (D) $\frac{(\sqrt{3}+1)}{2}$

(6) If θ is acute angle and given that $\cot q = \frac{7}{8}$ then the value of $\frac{(1+\sin q)(1-\sin q)}{(1+\cos q)(1-\cos q)}$ is equal to

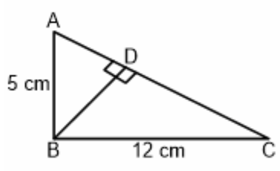
(A) $\frac{49}{64}$ (B) $\frac{64}{49}$

(C) $\frac{1}{49}$ (D) $\frac{1}{64}$

(7) In figure BD is perpendicular to side AC then the value of BD is

(A) $\frac{60}{13}$ (B) $\frac{65}{12}$

(C) $\frac{13}{5}$ (D) None of these



(8) In above situation the value of $\sin\angle BAC$ is

(A) $\frac{5}{12}$ (B) $\frac{12}{13}$

(C) $\frac{5}{12}$ (D) None of these

(9) If $\cos\theta = \frac{2}{3}$, then the value of $2\sec^2\theta + 2\tan^2\theta - 9$ is

(A) 2 (B) -2

(C) 1 (D) 3

(10) In $\triangle ABC$, right angled at B. The value of $\sin A \cdot \sec C$ is equal to

(A) 2 (B) 3

(C) 1 (D) None of these

(11) The value of $\sin 30^\circ \cdot \cos 60^\circ + \cos 30^\circ \sin 60^\circ$ is equal to

(A) $\frac{\sqrt{3}+1}{2}$ (B) 1

(C) $\frac{\sqrt{3}+1}{4}$ (D) None of these

(12) $\frac{1-\tan^2 45^\circ}{1+\tan^2 45^\circ} =$
(A) $\tan 90^\circ$ (B) $\sin 45^\circ$
(C) 1 (D) 0

(13) $\sin 2A = 2\sin A$ is true when $A =$
(A) 0° (B) 30°
(C) 45° (D) None of these

(14) $\frac{2\tan 30^\circ}{1-\tan^2 30^\circ} =$
(A) $\sin 60^\circ$ (B) $\cos 60^\circ$
(C) $\tan 60^\circ$ (D) $\sin 30^\circ$

(15) The value of $\frac{\sin 30^\circ + \tan 45^\circ - \operatorname{cosec} 60^\circ}{\sec 30^\circ + \cos 60^\circ + \cot 45^\circ}$ is equal to
(A) $\frac{4\sqrt{3}-9}{11}$ (B) $\frac{9-4\sqrt{3}}{-11}$
(C) $\frac{-\sqrt{3}}{4+3\sqrt{3}}$ (D) none of these

(16) The value of $\frac{5\cos^2 60^\circ + 4\sec^2 30^\circ - \tan^2 45^\circ}{\sin^2 30 + \cos^2 30^\circ}$ is equal to
(A) $\frac{5}{12}$ (B) $\frac{1}{4}$
(C) $\frac{67}{12}$ (D) $\frac{3}{4}$

(17) When $x = 90^\circ$ then $\tan x$ is
(A) Infinity (B) zero
(C) Not defined (D) 1

(18) The value of $\cos 48^\circ - \sin 42^\circ$ is equal to
(A) 0 (B) 1
(C) 2 (D) None of these

(19) The value of $\frac{\tan 64^\circ}{\cot 26^\circ}$ is equal to
(A) 0 (B) 1
(C) 2 (D) None of these

(20) The value of $\tan 48^\circ \tan 23^\circ \tan 42^\circ \tan 67^\circ$ is equal to
(A) 0 (B) 1
(C) 2 (D) None of these

(21) The value of $\cos 38^\circ \cos 52^\circ - \sin 38^\circ \sin 52^\circ$ is equal to
(A) 0 (B) 1
(C) 2 (D) None of these

(22) If $\tan 2A = \cot(A - 18^\circ)$, where $2A$ is an ancute angle, then value of A
(A) 36° (B) 18°
(C) 72° (D) None of these

(23) If $\tan A = \cot B$ then
(A) $A = B$ (B) $A + B = 45^\circ$
(C) $A + B = 90^\circ$ (D) None of these

(24) If $\sec 4A = \operatorname{cosec}(A - 20^\circ)$ where $4A$ is an acute angle, then value of A
(A) 22° (B) 18°
(C) 24° (D) None of these

(25) The value of $\tan 1^\circ \tan 2^\circ \tan 3^\circ \ldots . \tan 45^\circ \ldots . \tan 88^\circ \tan 89^\circ$ is equal to
(A) 0 (B) 1
(C) 2 (D) None of these

Answer Key

OBJECTIVE TYPE QUESTION

(1)	**(2)**	**(3)**	**(4)**	**(5)**	**(6)**	**(7)**	**(8)**	**(9)**	**(10)**
(C)	(C)	(A)	(A)	(B)	(A)	(A)	(B)	(B)	(C)
(11)	**(12)**	**(13)**	**(14)**	**(15)**	**(16)**	**(17)**	**(18)**	**(19)**	**(20)**
(B)	(D)	(A)	(C)	(D)	(C)	(C)	(A)	(B)	(B)
(21)	**(22)**	**(23)**	**(24)**	**(25)**					
(A)	(A)	(C)	(A)	(B)					