QUADRATIC EQUATIONS

Concepts Covered

- Introduction to Quadratic Equations
- Factorization Method
- Solving a Quadratic Equation
- Roots of a Quadratic Equation
- Nature of Roots

Introduction

Very often we come across many equations involving several powers of one variable. If the indices of all these powers are integers then the equation is called a polynomial equation. If the highest index of a polynomial equation in one variable is two, then it is a **Quadratic Equation**.

A quadratic equation is a **second-degree** polynomial in variable $\mathbf{x}$ usually equated to zero. In other words, for an equation to be quadratic, the **coefficient of** x^2 should **not** be **zero** and the coefficients of any higher power of x should be 0.

General form of a quadratic equation is $\boxed{ax^2 + bx + c = 0}$, where a $\neq 0$ (and a, b, c are real numbers).

The following are some quadratic equations.

1. $x^2 - 5x + 6 = 0$ a = 1
2. $x^2 - x - 6 = 0$ a = 1
3. $4x^2 + 3x - 2 = 0$ a = 4
4. $2x^2 + x - 3 = 0$ a = 2

Example:

Check whether the following are quadratic equations.

i. $\mathbf{(3x-2)(x-3) = (x+5)(x-1)}$
ii. $\mathbf{(x+3)^3 = x^3 + 20}$
iii. $\mathbf{2x + \frac{3}{x} = 5, x \neq 0}$

Solution:

i. Given,

$$(3x-2)(x-3) = (x+5)(x-1)$$
$$3x^2 - 9x - 2x + 6 = x^2 - x + 5x - 5$$
$$3x^2 - x^2 - 11x - 4x + 6 + 5 = 0$$
$$\mathbf{2x^2 - 15x + 11 = 0}$$

which is of the form $ax^2 + bx + c = 0$.

Therefore,
$(3x-2)(x-3) = (x+5)(x-1)$ is a quadratic equation.

ii. Given,

$$(x+3)^3 = x^3 + 20$$
$$x^3 + 27 + 9x(x+3) = x^3 + 20$$
$$x^3 + 27 + 9x^2 + 27x = x^3 + 20$$
$$\mathbf{9x^2 + 27x + 7 = 0}$$

which is of the form $ax^2 + bx + c = 0$.

Therefore, $(x+3)^3 = x^3 + 20$ is a quadratic equation.

iii. Given,

$$2x + \frac{3}{x} = 7, x \neq 0$$
$$2x^2 + 3 = 7x$$
$$\mathbf{2x^2 - 7x + 3 = 0}$$

Therefore
, $2x + \frac{3}{x} = 7$, can be reduced to a quadratic equation

Example:

Check whether the following are quadratic equations:

(i) $\mathbf{(2x-1)(x-3) = (x+4)(x-2)}$
(ii) $\mathbf{(x+2)^3 = 2x(x^2-1)}$
(iii) $\mathbf{(x+1)^3 = x^3 + x + 6}$
(iv) $\mathbf{x(x+3) + 6 = (x+2)(x-2)}$

Solution:

(i) $(2x-1)(x-3) = (x+4)(x-2)$

$\Rightarrow 2x^2 - 7x + 3 = x^2 + 2x - 8$

$\Rightarrow \mathbf{x^2 - 9x + 11 = 0.}$

This is of the form $ax^2 + bx + c = 0$,

Here, $a = 1$, $b = -9$ and $c = 11$.

Hence, the given equation is a quadratic equation.

(ii) $(x+2)^3 = 2x(x^2-1)$

$\Rightarrow x^3 + 8 + 6x(x+2) = 2x^3 - 2x$

$\Rightarrow x^3 + 6x^2 + 12x + 8 = 2x^3 - 2x$

$\Rightarrow \mathbf{x^3 - 6x^2 - 14x - 8 = 0}$

This is not of the form $ax^2 + bx + c = 0$.

Hence, the given equation is not quadratic.

(iii) $(x+1)^3 = x^3 + x + 6$

$\Rightarrow x^3 + 1 + 3x(x+1) = x^3 + x + 6$

$\Rightarrow \mathbf{3x^2 + 2x - 5 = 0}$

This is of the form $ax^2 + bx + c = 0$,

where $a = 3, b = 2$ and $c = -5$.

Hence, the given equation is a quadratic equation.

(iv) $x(x+3) + 6 = (x+2)(x-2)$

$\Rightarrow x^2 + 3x + 6 = x^2 - 4$

$\Rightarrow \mathbf{3x + 10 = 0}$

This is not of the form $ax^2 + bx + c = 0$.

Hence, the given equation is not a quadratic equation.

Example:

The perimeter of a rectangle is 20 cm and its area is $24\ \text{cm}^2$. Formulate the quadratic equation to determine the length and breadth of the plot.

Solution:

Let the length of the rectangle be $\mathbf{x\ cm}$.

Perimeter = 2 (length + breadth) Then the breadth of the rectangle is $\left(\frac{20-2x}{2}\right)\text{cm} = \mathbf{(10-x)cm}$

Now, area of the rectangle $= 24\ \text{cm}^2$. Therefore,

$x(10-x) = 24$

$10x - x^2 = 24$

$\boxed{\mathbf{x^2 - 10x + 24 = 0}}$

This is the required quadratic equation

Example:

Two consecutive positive odd numbers are such that the sum of their squares is 130. Formulate the quadratic equation whose roots are these numbers.

Solution:

Let two consecutive positive odd numbers be $\mathbf{x}$ and $\mathbf{x+2}$.

Sum the square of the numbers $= x^2 + (x+2)^2$

It is given that the sum is 130.

Therefore,

$x^2 + (x+2)^2 = 130$

$x^2 + x^2 + 4x + 4 = 130$

$2x^2 + 4x - 126 = 0$

$\boxed{\mathbf{x^2 + 2x - 63 = 0}}$

This is the required quadratic equation.

Solving A Quadratic Equation by Factorisation

Let the given quadratic equation be $\mathbf{ax^2 + bx + c = 0}$, where $a \neq 0$. Let $(ax^2 + bx + c)$ be expressible as the product of two linear expressions, say $(px+q)$ and $(rx+s)$ where p, q, r, s are real numbers such that $p \neq 0$ and $r \neq 0$. Then,

$ax^2 + bx + c = 0$

$\Rightarrow (px+q)(rx+s) = 0$

$\Rightarrow (px+q) = 0$ or $(rx+s) = 0$

$\Rightarrow x = -\frac{q}{p}$ or $x = -\frac{s}{r}$

Two possible values of ‘x’ are there which satisfy the given quadratic equation.

Example:

Solve : $\mathbf{(x+2)(3x-5) = 0.}$

Solution: We have

$(x+2)(3x-5) = 0$

$\Rightarrow x + 2 = 0$ or $3x - 5 = 0$

$\Rightarrow x = -2$ or $x = \frac{5}{3}$

Hence, the roots of the given equation are -2 and $\frac{5}{3}$

Example:

Solve : $\mathbf{5x^2 - 8x = 0.}$

Solution: We have

$5x^2 - 8x = 0$

$\Rightarrow x(5x-8) = 0$

$\Rightarrow x = 0$ or $5x - 8 = 0$

$\boxed{\Rightarrow x = 0 \text{ or } x = \frac{8}{5}}$

Hence, $\mathbf{0}$ and $\frac{8}{5}$ are the roots of the given equation.

Factorization by Splitting the Middle Term

In Class IX, we have learned how to factorise a quadratic polynomial by splitting the middle term. We shall use it for finding the roots of a quadratic equation.

To factorize a quadratic polynomial, we split the coefficient of x into two numbers such that their product is equal to the product of the coefficient of x^2 and the constant term and sum is equal to the coefficient of x.
Factorization of polynomials of the form $ax^2 + bx + c$.
Step 1: Take the product of the constant term and the coefficient of x^2, i.e., ac.
Step 2: Now this product ac is to split into two factors m and n such that m + n is equal to the coefficient of x, i.e., b.
Step 3: Then we pair one of them, say mx, with ax^2 and the other nx, with c and factorize.

Example:

Solve: $6x^2 - x - 2 = 0$ by the factorisation method.
Solution: We write, $-x = -4x + 3x$ as $(-4x) \times 3x = -12x^2 = 6x^2 \times (-2)$.

$\therefore 6x^2 - x - 2 = 0$
$\Rightarrow 6x^2 - 4x + 3x - 2 = 0 \Rightarrow 2x(3x - 2) + (3x - 2) = 0$
$\Rightarrow (3x - 2)(2x + 1) = 0 \Rightarrow 3x - 2 = 0$ or $2x + 1 = 0$
$\Rightarrow x = \frac{2}{3}$ or $x = \frac{-1}{2}$.
Hence, $\frac{2}{3}$ and $\frac{-1}{2}$ are the roots of the given equation.

Example:

Solve: $8x^2 - 22x - 21 = 0$ by the factorisation method.
Solution: We write,

$-22x = -28x + 6x$ as $8x^2 \times (-21) = -168x^2 = (-28x) \times 6x$.
$\therefore 8x^2 - 22x - 21 = 0$
$\Rightarrow 8x^2 - 28x + 6x - 21 = 0 \Rightarrow 4x(2x - 7) + 3(2x - 7) = 0$
$\Rightarrow (2x - 7)(4x + 3) = 0 \Rightarrow 2x - 7 = 0$ or $4x + 3 = 0$
$\Rightarrow x = \frac{7}{2}$ or $x = \frac{-3}{4}$.
Hence, $\frac{7}{2}$ and $\frac{-3}{4}$ are the roots of the given equation.

Example:

Solve: $6x^2 + 40 = 31x$.
Solution: The given equation may be written as $6x^2 - 31x + 40 = 0$. We write,

$-31x = -16x - 15x$ as $6x^2 \times 40 = 240x^2 = (-16x) \times (-15x)$.
$\therefore 6x^2 - 31x + 40 = 0$
$\Rightarrow 6x^2 - 16x - 15x + 40 = 0 \Rightarrow 2x(3x - 8) - 5(3x - 8) = 0$
$\Rightarrow (3x - 8)(2x - 5) = 0 \Rightarrow 3x - 8 = 0$ or $2x - 5 = 0$
$\Rightarrow x = \frac{8}{3}$ or $x = \frac{5}{2}$
Hence, $\frac{8}{3}$ and $\frac{5}{2}$ are the roots of the given equation.

Example:

Solve:
(i) $\sqrt{x} + 2x = 1$
Solution:

$\sqrt{x} + 2x = 1 \Rightarrow \sqrt{x} = 1 - 2x$
$\Rightarrow x = 1 + 4x^2 - 4x$
i.e., $1 + 4x^2 - 4x - x = 0$
$\Rightarrow 4x^2 - 5x + 1 = 0$ i.e., $4x^2 - 4x - x + 1 = 0$
$\Rightarrow 4x(x - 1) - 1(x - 1) = 0$

i.e., $(x - 1)(4x - 1) = 0$
$\Rightarrow x - 1 \;\; 0$
or $4x - 1 = 0$
i.e., $x = 1$ or $x = \frac{1}{4}$

Example:

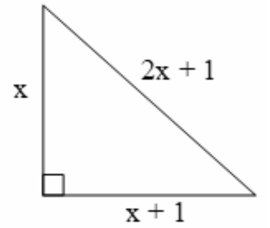
Divide 16 into two parts such that twice the square of the larger part exceeds the square of the smaller part by 164.
Solution: Let larger part be x, therefore the smaller part $= 16 - x$

Given : $2x^2 - (16 - x)^2 = 164$
$\Rightarrow 2x^2 - (256 + x^2 - 32x) - 164 = 0$
i.e. $2x^2 - 256 - x^2 + 32x - 164 = 0$
$\Rightarrow x^2 + 32x - 420 = 0$
On factorizing, it gives : $(x + 42)(x - 10) = 0$
i.e., $x = -42$ or $x = 10$
$\therefore x = 10$
Hence the larger part $= 10$ and the smaller part $= 16 - x = 16 - 10 = 6$

Example:

The hypotenuse of a right triangle is $1\ m$ less than twice the shortest side. If the third side is $1\ m$ more than the shortest side, find the sides of the triangle.

Solution: Let the shortest side be x m.



$\therefore$ Hypotensuse $= (2x-1)m$ and the third side $= (x+1)m$

Applying Pythagoras theorem, we get ;

$(2x-1)^2 = x^2 + (x+1)^2$

$\Rightarrow 4x^2 - 4x + 1 = x^2 + x^2 + 2x + 1$

i.e.,$2x^2 - 6x = 0$

$\Rightarrow x^2 - 3x = 0$

i.e., $x(x-3) = 0 \Rightarrow x = 0$ or $x = 3$

Since, $x = 0$ makes the triangle impossible. therefore, $x = 3$

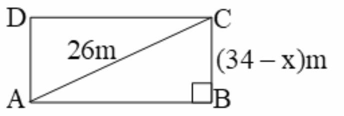
And, sides of the triangle are $= x, 2x - 1$ and $x + 1$

putting x = 3, sides of the triangle will come 3m, 4m and 5m.

Example:

If the perimeter of a rectangular plot is $68\ m$ and its diagonal is $26\ m$. Find its area.

Solution: Let the length of plot $= x$ m



$\because$ 2(length + breadth) = perimeter

$\Rightarrow 2(x +$ breadth $) = 68$

$\Rightarrow$ $x +$ breadth $= \dfrac{68}{2}$ and breadth $= (34-x)m$

Given its diagonal $= 26$ m and we know each angle of the rectangle $= 90°$.

$\therefore x^2 + (34-x)^2 = 26^2$

[Applying Pythagoras Theorem]

$\Rightarrow x^2 + 1156 - 68x + x^2 - 676 = 0$

$\Rightarrow 2x^2 - 68x + 480 = 0$

$\Rightarrow x^2 - 34x + 240 = 0$

i.e., $x^2 - 34x + 240 = 0$

On factorising, we get : $(x-24)(x-10) = 0$

i.e., $x = 24$ or $x = 10$

$x = 24$

$\Rightarrow$ length $= 24$ m and breadth

$= (34-24)m = 10$ m

and, $x = 10$

$\Rightarrow$ length $= 10$ m and breadth

$= (34-10)m = 24$ m

$\therefore$ Dimensions of the given rectangular plot are

24 m and 10 m.

Hence, its area = length $\times$ breadth $= 24\ m \times$

$10\ m = 240\ m^2$

Example:

One-fourth of a herd of camels was seen in the forest. Twice the square root of the herd had gone to mountains and the remaining 15 camels were seen on the bank of a river. Find the total number of camels. **[HOTS]**

Solution : Let x be the total number of camels.

Then, the number of camels in the forest $= \frac{x}{4}$

Number of camels on mountains $= 2\sqrt{x}$

and number of camels on the bank of river $= 15$

Thus, the total number of camels $= \frac{x}{4} + 2\sqrt{x} + 15$

Now by hypothesis, we have

$\frac{x}{4} + 2\sqrt{x} + 15 = x \Rightarrow 3x - 8\sqrt{x} - 60 = 0$

Let $\sqrt{x} = y$, then $x = y^2$

$3y^2 - 8y - 60 = 0 \Rightarrow 3y^2 - 18y + 10y - 60 = 0$

$3y(y-6) + 10(y-6) = 0 \Rightarrow (3y+10)(y-6) = 0$

$y = 6$ or $y = -\dfrac{10}{3}$

Now, $y = -\dfrac{10}{3} \Rightarrow x = \left(-\dfrac{10}{3}\right)^2 = \dfrac{100}{9}$ $(\because x = y^2)$

But the number of camels cannot be a fraction.

$\therefore$ $y = 6 \Rightarrow x = 6^2 = 36$

Hence, the number of camels $= 36$

Example:

Two taps running together can fill a tank in $2\frac{6}{11}$ hours. If one tap takes 3 hours more than the other to fill the tank, then how much time will each tap take to fill the tank?

[HOTS]

Solution: Let, the time taken by faster tap to fill the tank be x hours.

Therefore, time taken by slower tap to fill the $\text{tank} = (x+3)$ hours

Since the faster tap takes x hours to fill the tank.

∴ The portion of the tank filled by the faster tap in one hour $= \frac{1}{x}$

The portion of the tank filled by the slower tap in one hour $= \frac{1}{x+3}$

The portion of the tank filled by the two taps together in one hour $= \frac{1}{\frac{28}{11}} = \frac{11}{28}$

According to question,

$\Rightarrow \frac{1}{x} + \frac{1}{x+3} = \frac{11}{28} \Rightarrow \frac{x+3+x}{x(x+3)} = \frac{11}{28}$

$28(2x+3) = 11x(x+3) \Rightarrow 56x + 84 = 11x^2 + 33x$

$11x^2 - 23x - 84 = 0 \Rightarrow 11x^2 - 44x + 21x - 84 = 0$

$11x(x-4) + 21(x-4) = 0 \Rightarrow (x-4)(11x+21) = 0$

$x - 4 = 0$ or $11x + 21 = 0$

$x = 4$ or $x = \frac{-21}{11}$

$x = 4$

Hence, time taken by faster tap to fill the $\text{tank} = x = 4$ hours and time taken by slower tap $= x + 3 = 4 + 3 = 7$ hours.

Example:

Directions: In the following questions, A statement of Assertion (A) is followed by a statement of Reason (R). Mark the correct choice as.

Assertion (A): The positive root of $\sqrt{3x^2+6} = 9$ is 5.

Reason (R): If $x = \frac{2}{3}$ and $x = -3$ are roots of the quadratic equation $ax^2 + 7x + b = 0$, then the value of a and b are 3 and -6.

Solution: In case of assertion:

$\sqrt{3x^2+6} = 9$

$3x^2 + 6 = 81$

$3x^2 = 81 - 6 = 75$

$x^2 = \frac{75}{3} = 25$

$x = \pm 5$

$\Rightarrow 4a + 42 + 9b = 0$

$\Rightarrow 4a + 9b = -42$ ... (i)

again, substituting $x = -3$

$\Rightarrow 9a + b = 21$ ... (ii)

Solving (i) and (ii), we get

$a = 3$ and $b = -6$

∴ Reason is correct

Hence, both assertion and reason are correct but the reason is not the correct explanation for assertion.

Example:

Directions: In the following questions, A statement of Assertion (A) is followed by a statement of Reason (R). Mark the correct choice as.

Assertion (A): If the coefficient of x^2 and the constant term have the same sign and if the coefficient of x term is zero then the quadratic equation has no real roots.

Reason (R): The equation $13\sqrt{3}x^2 + 10x + \sqrt{3} = 0$

Solution: In case of assertion:

Since, in this case discriminant is always negative, so it has no real roots, that is, if $b = 0$, then $b^2 - 4ac \Rightarrow -4ac < 0$ and $ac > 0$.

∴ Assertion is correct.

In case of reason:

Here, $a = 13\sqrt{3}, b = 10$ and $c = \sqrt{3}$

Then,

$b^2 - 4ac = (10)^2 - 4(13\sqrt{3})(\sqrt{3})$

$= 100 - 156$

$= -56$

As $D < 0$.

So, the equation has no real roots.

∴ The reason is correct.

Hence, assertion and reason are correct and reason is the correct explanation for assertion.

Example:

Directions: In the following questions, A statement of Assertion (A) is followed by a statement of Reason (R). Mark the correct choice as.

Assertion (A): Every quadratic equation has exactly one root.
Reason (R): Every quadratic equation has almost two roots.
Solution: In case of assertion:

Since, a quadratic equation has two and only two roots.
∴ Assertion is incorrect.
In case of reason:
Because every quadratic polynomial has almost two roots.
∴ Reason is correct.
Hence, assertion is incorrect but reason is correct.

(1) Represent the following situations in the form of quadratic equations:

(i) Rohan's mother is 26 years older than him. The product of their ages (in years) 3 years from now will be 360. We would like to find Rohan's present age.

(ii) A train travels a distance of 480 km at a uniform speed. If the speed had been 8 km/h less, then it would have taken

(2) If α and β are the roots of the equation $x^2 - 6x + 8 = 0$, then find the values of

(i) $\alpha^2 + \beta^2$

(ii) $\frac{1}{\alpha} + \frac{1}{\beta}$

(iii) $\alpha - \beta (\alpha > \beta)$

(3) Find the roots of the following quadratic equations by the factorisation method:

(i) $2x^2 + \frac{5}{3}x - 2 = 0$

(ii) $\frac{2}{5}x^2 - x - \frac{3}{5} = 0$

(iii) $x = -\frac{1}{2}$ or $x = 3$

(4) Form the quadratic equation whose roots are less by "1" than those of $3x^2 - 4x - 1 = 0$

(5) If the Quadratic equation $Px^2 - 2\sqrt{5}Px + 15 = 0$ has two equal roots then find the value of P.

(6) Solve for x by factorisation

(a) $8x^2 - 22x - 21 = 0$

(b) $3\sqrt{5}x^2 + 25x + 10\sqrt{5} = 0$

(c) $3x^2 - 2\sqrt{6}x + 2 = 0$

Answer Key

(1) (i) $x^2 + 32x - 273 = 0$ (ii) $x^2 - 8x - 1280 = 0$

(2) (i) 20 (ii) $3/4$ (iii) 2

(3) (i) $x = -\frac{3}{2}$ or $x = \frac{2}{3}$

(4) $3x^2 + 2x - 2 = 0$

(5) p = 3

(6) (a) $x = \frac{7}{2}, x = -\frac{3}{4}$

(b) $x = \sqrt{5}, x = \frac{-2\sqrt{5}}{3}$

(c) $x = \frac{\sqrt{2}}{3}, x = \frac{\sqrt{2}}{3}$

The Quadratic Formula

The method of completing the square always works. By applying it to the general quadratic equation $ax^2 + bx + c = 0$ we obtain the well-known quadratic formula. To derive the formula, we will begin by multiplying the equation through by $4a$, which although not the usual first step in completing the square, will make the algebra much easier.

Multiplying by '4a'

$\mathbf{4a^2x^2 + 4abx + 4ac = 0}$

We now note that $(2ax + b)^2 = 4a^2x^2 + 4abx + b^2$ so adding b^2 will produce a square.

$4a^2x^2 + 4abx = -4ac$

$4a^2x^2 + 4abx + b^2 = b^2 - 4ac$

$\mathbf{(2ax + b)^2 = b^2 - 4ac}$.

$\mathbf{ax^2 + bx + c = 0}$

We pause at this stage to note that if $b^2 - 4ac$ is negative, then there is no solution. If $b^2 - 4ac$ is positive, we then proceed to take the positive and negative square roots to solve for x. If $b^2 - 4ac$ is equal to 0, then there will only be 1 solution. We suppose then that $b^2 - 4ac$ is positive and proceed to find the solutions.

$(2ax + b)^2 = b^2 - 4ac$

$2ax + b = b^2 - 4ac$ or $2ax + b = -\sqrt{b^2 - 4ac}$

$$\mathbf{x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ or } x = -\frac{-b - \sqrt{b^2 - 4ac}}{2a}}$$

This last formula is called the quadratic formula, sometimes written as $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

<u>Special case</u>: If the quantity $b^2 - 4ac = 0$ then there will only be one solution, $x = -\frac{b}{2a}$

In this case, the quadratic will be a perfect square.

The quantity $b^2 - 4ac$ plays an important role in the theory of quadratic equations and is called the discriminant.

Thus, in summary, when solving $ax^2 + bx + c = 0$, first calculate the discriminant $b^2 - 4ac$. Then,

- If $b^2 - 4ac$ is negative, then there is no solution.
- If $b^2 - 4ac$ is positive, then the solutions are $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, x = -\frac{-b - \sqrt{b^2 - 4ac}}{2a}$
- If $b^2 - 4ac$ is zero, then there is only one solution $x = -\frac{b}{2a}$.

While students do not need to learn the derivation of the formula, they do need to remember the formula itself.

Note: If $b^2 - 4ac$ is zero, then the quadratic is a perfect square.

Example:

Find the roots of the quadratic equations by applying the quadratic formula.

(i) $2x^2 - 7x + 3 = 0$

Solution:

On comparing the given equation with $ax^2 + bx + c = 0$, **we get, $a = 2, b = -7$ and $c = 3$**

By using the quadratic formula, we get,

$\Rightarrow x = (7 \pm \sqrt{49 - 24})/4 \Rightarrow x = (7 \pm \sqrt{25})/4$

$\Rightarrow x = (7 \pm 5)/4 \Rightarrow x = (7 + 5)/4$ or $x = (7 - 5)/4$

$\Rightarrow x = 12/4$ or $2/4$

$\therefore x = 3$ or $\frac{1}{2}$

(ii) $2x^2 + x - 4 = 0$

Solution: On comparing the given equation with $ax^2 + bx + c = 0$, we get, $a = 2, b = 1$ and $c = -4$

By using the quadratic formula, we get,

$\Rightarrow x = (-1 \pm \sqrt{1} + 32)/4 \Rightarrow x = (-1 \pm \sqrt{33})/4$

$\therefore x = (-1 + \sqrt{33})/4$ or $x = (-1 - \sqrt{33})/4$

Example:

Solve the equation $\frac{1}{2x-5} + \frac{5}{2x-1} = 2$ by using Quadratic formula.

Solution:

$\frac{1}{2x-5} + \frac{5}{2x-1} = 2$

Multiplying throughout by $(2x - 5)(2x - 1)$, we get

$(2x - 1) + 5(2x - 5) = 2(2x - 5)(2x - 1)$

$2x - 1 + 10x - 25 = 8x^2 - 24x + 10$

$8x^2 - 36x + 36 = 0$

$2x^2 - 9x + 9 = 0$

Comparing this equation with General Quadratic Equation

Here, $a = 2, b = -9, c = 9$

Putting these values in the Quadratic formula

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{(-9) \pm \sqrt{(-9)^2 - 4(2)(9)}}{2(2)} = \frac{9 \pm \sqrt{81 - 72}}{4}$

$= \frac{9 \pm 3}{4}$

$x = \frac{9+3}{4}$ or $x = \frac{9-3}{4}$

$x = \frac{12}{4}$ or $x = \frac{6}{4}$

$x = 3$ or $x = -\frac{3}{2}$

Example:

Solve:

$\sqrt{3x^2-2}+1=2x$ **by Quadratic Formula**

Solution: $\because \sqrt{3x^2-2}+1=2x$

$\sqrt{3x^2-2}=2x-1$

$(3x^2-2)=(2x-1)^2$

$3x^2-2=4x^2+1-4x$

$4x^2-3x^2-4x+1+2=0$

$x^2-4x+3=0$

$\Rightarrow x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$=\dfrac{-(-4)\pm\sqrt{(-4)^2-4(1)(3)}}{2(1)}$

$=\dfrac{4\pm\sqrt{16-12}}{2}$

$=\dfrac{4\pm\sqrt{4}}{2}$

$=\dfrac{4\pm 2}{2}$

$\therefore x=\dfrac{4+2}{2},\dfrac{4-2}{2}$

$x=3,1$

Example:

Solve:

$\sqrt{2x^2+19}+x=13$

Solution : $\sqrt{2x^2+19}+n=13$

$\sqrt{2x^2+19}=13-x$

$2x^2+19=(13-x)^2$

$2x^2+19=169+x^2-26x$

$x^2+26x+150=0$

$x=-\dfrac{b\pm\sqrt{b^2-4ac}}{2a}$

$x=\dfrac{-26\pm\sqrt{636-4(150)}}{2}$

$x=\dfrac{-26\pm\sqrt{36}}{2}$

$x=\dfrac{-26\pm 6}{2}$

$x=-13\pm 3$

$x=-10, \quad -16$

Example:

Solve the following quadratic equation: $9x^2-9(a+b)x+[2a^2+5ab+2b^2]=0$ **[HOTS]**

Solution : Consider the equation $9x^2-9(a+b)x+[2a^2+5ab+2b^2]=0$

Now comparing with $Ax^2+Bx+C=0$, we get : $A=9, B=-9(a+b)$ and $C=[2a^2+5ab+2b^2]$

Now discriminant,

$D=B^2-4AC$

$=\{-9(a+b)\}^2-4\times 9(2a^2+5ab+2b^2)=9^2(a+b)^2-4\times 9(2a^2+5ab+2b^2)$

$=9\{9(a+b)^2-4(2a^2+5ab+2b^2)\}=9\{9a^2+9b^2+18ab-8a^2-20ab-8b^2\}$

$=9\{a^2+b^2-2ab\}=9(a-b)^2$

Now using the quadratic formula,

$x=\dfrac{-B\pm\sqrt{D}}{2A}$, we get $x=\dfrac{9(a+b)\pm\sqrt{9(a-b)^2}}{2\times 9}$

$\Rightarrow x=\dfrac{9(a+b)\pm 3(a-b)}{2\times 9}\Rightarrow x=\dfrac{3(a+b)\pm(a-b)}{6}$

$\Rightarrow x=\dfrac{(3a+3b)+(a-b)}{6}$ and $x=\dfrac{(3a+3b)-(a-b)}{6}$

$\Rightarrow x=\dfrac{(4a+2b)}{6}$ and $x=\dfrac{(2a+4b)}{6}$

$\Rightarrow x=\dfrac{2a+b}{3}$ and $x=\dfrac{a+2b}{3}$ are required solutions.

Example:

In a rectangular park of dimensions $50\text{ m}\times 40\text{ m}$, a rectangular pond is constructed so that the area of grass strip of uniform width surrounding the pond would be 1184m^2. Find the length and breadth of the pond.

[HOTS NCERT Exemplar]

Solution: Let ABCD be a rectangular lawn and EFGH be a rectangular pond. Let x m be the width of grass area, which is same around the pond.

$\Rightarrow$ Area of rectangular lawn - area of pond $=1184\text{ m}^2$

$\Rightarrow 50\times 40-\{(50-2x)\times(40-2x)\}=1184$

$\Rightarrow 2000-(2000-80x-100x+4x^2)=1184$

$\Rightarrow 2000-2000+180x-4x^2=1184$

$\Rightarrow 4x^2-180x+1184=0 \Rightarrow x^2-45x+296=0$

$\Rightarrow x^2-37x-8x+296=0 \Rightarrow x(x-37)-8(x-37)=0$

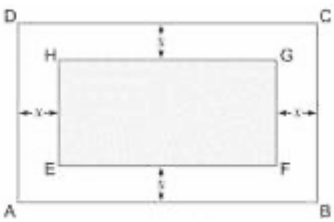
$\Rightarrow (x-37)(x-8)=0 \Rightarrow x-37=0$ or $x-8=0$

$\Rightarrow x=37$ or $x=8$

$x=37$ is not possible as in this case length of the pond becomes $50-2\times 37=-24$ (not possible)

Hence, $x=8$ is acceptable

$\therefore$ Length of pond $=50-2\times 8=34$ m Breadth of pond $=40-2\times 8=24$ m



Example:

The diagonal of a rectangular field is 60 metres more than the shorter side. If the longer side is 30 metres more than the shorter side, find the sides of the field. [HOTS NCERT]

Solution: Let $ABCD$ be the rectangular field. Let the shorter side BC of the rectangle $= x$ metres.

According to the question,

Diagonal of the rectangle, $AC = (x + 60)$ metres

side of the rectangle, $AB = (x + 30)$ metres

By Pythagoras theorem, $AC^2 = AB^2 + BC^2$

$\therefore\ (x+60)^2 = (x+30)^2 + x^2$

or $x^2 + 120x + 3600 = x^2 + 60x + 900 + x^2$

$\therefore\ (2x^2 - x^2) + (60x - 120x) + 900 - 3600 = 0$

or

$x^2 - 60x - 2700 = 0$

$(x - 90)(x + 30) = 0$

or

Either $x - 90 = 0$ or $x + 30 = 0$

$\Rightarrow\ x = 90$ or $x = -30$ (But side cannot be negative)

So, the shorter side of rectangle $= 30\ \text{m}$

and longer side of rectangle $= 130\ \text{m}$

Example:

Read the following text and answer the following questions on the basis of the same:
Akshay and Shruti are having fun with the marbles game. They have a total of 50 marbles. Both of them have dropped 5 marbles, bringing their total number of marbles to 351.

(1) If Akshay had x number of marbles, then the number of marbles Shruti had:

(A) $x - 50$ (B) $50 - x$

(C) $50x$ (D) $x - 5$

Ans. Option (B) is correct.

Solution: If Akshay had x number of marbles, then Shruti had $(50 - x)$ marbles, because there are total of 45 marbles.

(2) Number of marbles left with Shruti, when she lost 5 marbles:

(A) $x - 55$ (B) $45 - x$

(C) $55 - x$ (D) $x - 45$

Ans. Option (B) is correct.

Solution: Number of marbles left with Shruti,

when she lost 5 marbles $= (50 - x - 5)$

$= (45 - x)$

(3) The quadratic equation related to the given problem is:

(A) $x^2 - 45x + 324 = 0$ (B) $x^2 + 45x + 324 = 0$

(C) $x^2 - 45x - 324 = 0$ (D) $-x^2 - 45x + 324 = 0$

Ans. Option (A) is correct.

Solution: According to question,

$(x - 5)(45 - x) = 351$

$\Rightarrow\ -x^2 - 225 + 45x + 5x - 351 = 0$

$\Rightarrow x^2 - 50x + 576 = 0$

(4) Number of marbles Akshay had:

(A) 10 (B) 16

(C) 35 (D) 30

Ans. Option (B) is correct.

Solution: $45x - 225 - x^2 + 5x = 351$

$$x^2 - 50x + 576 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{4a}$$

$$= \frac{-(-50) \pm \sqrt{(50)^2 - 4(576)(1)}}{4(1)}$$

$$= \frac{50 \pm \sqrt{2500 - 2304}}{4} = \frac{50 \pm \sqrt{196}}{4} = \frac{50 \pm \sqrt{16}}{4}$$

$$= \frac{50 + 14}{4}, \quad \frac{50 - 14}{4}$$

$$x = 16, 9$$

(5) If Akshay had 9 marbles, then number of marbles Jivanti had:

(A) 10 (B) 41

(C) 36 (D) 35

Ans. Option (B) is correct.

Solution: If Akshay had 9 marbles, then Shruti had $(50 - 9) = 41$ marbles.

Nature of The Roots

Consider the expression $\mathbf{b^2 - 4ac}$. Since $b^2 - 4ac$ determines the nature of the roots of the quadratic equation, it is called the "**DISCRIMINANT**" of the quadratic equation. A quadratic equation has real roots only if $b^2 - 4ac \geq 0$. If $b^2 - 4ac < 0$, then the roots of the quadratic equation are complex conjugates.

The following table gives us a clear idea about the nature of the roots of a quadratic equation when a, b and c are all rational.

Condition for Discriminant	Nature of roots
When $b^2 - 4ac < 0$	The roots are complex conjugates *(we will learn more in Class 11)*
When $b^2 - 4ac = 0$	The roots are **rational** and **equal**.
When $b^2 - 4ac > 0$ and a **perfect square**	The roots are **rational** and **unequal**.
When $b^2 - 4ac > 0$ and **not a perfect square**	The roots are **irrational** and **unequal**.

Example:

Find the nature of the roots of the following equations. If the real roots exist, find them.

(i) $\mathbf{2x^2 - 6x + 3 = 0}$

(ii) $\mathbf{2x^2 - 3x + 5 = 0}$

Solution:

(i) The given equation $2x^2 - 6x + 3 = 0$

Comparing it with $ax^2 + bx + c = 0$, we get

$a = 2, b = -6$ and $c = 3$.

$\therefore$ Discriminant, $D = b^2 - 4ac = (-6)^2 - 4.2.3 = 36 - 24 = 12 > 0$ $\because$ $D > 0$, roots are real and unequal.

Now, $x = \frac{-b\pm\sqrt{D}}{2a} = \frac{6\pm\sqrt{12}}{2\times2} = \frac{6\pm2\sqrt{3}}{4} = \frac{3\pm\sqrt{3}}{2}$

Hence the roots are $x = \frac{3+\sqrt{3}}{2}, \frac{3-\sqrt{3}}{2}$

(ii) Here, the given equation is $2x^2 - 3x + 5 = 0$:

Comparing it with $ax^2 + bx + c = 0$, we get

$a = 2$, $b = -3$ and $c = 5$.

$\therefore$ Discriminant, $D = b^2 - 4ac = 9 - 4 \times 2 \times 5 = 9 - 40 = -31$

$\because$ $D < 0$, the equation has no real roots.

Example:

Find the value of k for each of the following quadratic equations, so that they have real and equal roots:

(i) $\mathbf{9x^2 + 18kx + 16 = 0}$

(ii) $\mathbf{(k + 1)x^2 - 2(k - 1)x + 1 = 0}$

Solution:

(i) The given equation $9x^2 + 8kx + 16 = 0$

Comparing it with $ax^2 + bx + c = 0$, we get

$a = 9$, $b = 8k$ and $c = 16$.

$\therefore$ Discriminant, $D = b^2 - 4ac = (8k)^2 - 4 \times 9 \times 16 = 64k^2 - 576$

Since roots are real and equal, so

Hence, $k = 3, -3$

(ii) The given equation $(k + 1)x^2 - 2(k - 1)x + 1 = 0$

Comparing it with $ax^2 + bx + c = 0$,

we get $a = (k + 1), b - 2(k - 1)$ and $c = 1$

$\therefore$ Discriminant, $D = b^2 - 4ac = 4(k - 1)^2 - 4(k + 1) \times 1$

$\Rightarrow 4(k^2 - 2k + 1) - 4k -$

$\Rightarrow 4k^2 - 8k + 4 - 4k - 4 = 4k^2 - 12k$

Since roots are real and equal, so

$D = 0 \Rightarrow 4k^2 - 12k = 0$

$\Rightarrow 4k(k - 3) = 0$

$\Rightarrow$ either $k = 0$ or $k - 3 = 0 \Rightarrow k = 0$ or $k = 3$

Hence, $\mathbf{k = 0, 3}$

(1) Find the nature of the roots of the following equations

(i) $2x^2 + 3x + 1 = 0$

(ii) $6x^2 = 7x + 5$

(iii) $3x^2 + 7x - 2 = 0$

(iv) $\sqrt{2}x^2 + 3x - \sqrt{8} = 0$

(2) For what value of K the roots of the given equations are equal.

(i) $x^2 + 3(K + 1)x + 4K + 5 = 0$

(ii) $x^2 + 2(K - 2)x - 8k = 0$

(iii) $(3K + 6)x^2 + 6x + K = 0$

(iv) $(K + 2)x^2 - 2Kx + K - 1 = 0$

(3) If the roots of $(c^2 - ab)x^2 - 2(a^2 - bc)x + (b^2 - ac) = 0$ are equal then prove that $a^3 + b^3 + c^3 = 3abc$

(4) For what value of ' K ' the equation (4-k) $x^2 + 2(k + 2)x + 8k + 1 = 0$ will be a perfect square.

(**Hint :** The equation will be perfect square if Disc. $b^2 - 4ac = 0$)

(5) Solve the following equations using quadratic formulae

(i) $\sqrt{3}x^2 + 10x + 7\sqrt{3} = 0$

(ii) $\sqrt{2}x^2 + 7x + 5\sqrt{2} = 0$

(iii) $(x - 1)^2 - 5(x - 1) - 6 = 0$

Answer Key

(1) (i) Real, rational, unequal

(ii) unequal, real and rational

(iii) irrational, unequal, real

(iv) Real, unequal, irrational

(2) (i) $1, \frac{-11}{9}$

(ii) -2

(iii) $1, -3$

(iv) 2

(3) 0,3

(4)

(5) (i) $x = -\sqrt{3}, x = \frac{-7\sqrt{3}}{3}$

(ii) $x = -\sqrt{2}, x = \frac{-5\sqrt{2}}{2}$

(iii) x = 0, y = 7

Exercise

OBJECTIVE TYPE QUESTION

(1) Which of the following is a quadratic equation?

(A) $x^2 + 2x + 1 = (4 - x)^2 + 3$
(B) $-2x^2 = (5 - x)\left(2x - \frac{2}{5}\right)$
(C) $(k + 1)x^2 + \frac{3}{2}x = 7$, where $k = -1$
(D) $x^3 - x^2 = (x - 1)^3$

(2) Which of the following is not a quadratic equation?

(A) $2(x - 1)^2 = 4x^2 - 2x + 1$
(B) $2x - x^2 = x^2 + 5$
(C) $(\sqrt{2}x + \sqrt{3})^2 + x^2 = 3x^2 - 5x$
(D) $(x^2 + 2x)^2 = x^4 + 3 + 4x^3$

(3) Which of the following equations has the sum of its roots as 3 ?

(A) $2x^2 - 3x + 6 = 0$
(B) $-x^2 + 3x - 3 = 0$
(C) $\sqrt{2}x^2 - \frac{3}{\sqrt{2}}x + 1 = 0$
(D) $3x^2 - 3x + 3 = 0$

(4) Which of the following equations has 2 as a root? **[CBSE 2012]**

(A) $x^2 - 4x + 5 = 0$
(B) $x^2 + 3x - 12 = 0$
(C) $2x^2 - 7x + 6 = 0$
(D) $3x^2 - 6x - 2 = 0$

(5) Which of the following quadratic expression can be expressed as a product of real linear factors?

(A) $x^2 - 2x + 3$
(B) $3x^2 - \sqrt{2x} - \sqrt{3}$
(C) $\sqrt{2x^2} - \sqrt{5x} + 3$
(D) None of these

(6) Two candidates attempt to solve a quadratic equation of the form $x^2 + px + q = 0$. One starts with a wrong value of p and finds the roots to be 2 and 6. The other starts with a wrong value of q and finds the roots to be 2 and -9. Find the correct roots of the equation :

(A) 3,4
(B) $-3, -4$
(C) $3, -4$
(D) $-3, 4$

(7) Solve for x: $15x^2 - 7x - 36 = 0$

(A) $\frac{5}{9}, -\frac{4}{3}$
(B) $\frac{9}{5}, -\frac{4}{3}$
(C) $\frac{5}{9} = -\frac{3}{4}$
(D) None of these

(8) Solve for y: $7y^2 - 6y - 13\sqrt{7} = 0$

(A) $\sqrt{7}, 2\sqrt{7}$
(B) $3, \frac{2}{\sqrt{7}}$
(C) $\frac{13}{\sqrt{7}}, -\sqrt{7}$
(D) None of these

(9) Solve for x: $6x^2 + 40x = 31$

(A) $\frac{3}{8}, \frac{2}{5}$
(B) $\frac{3}{8}, \frac{3}{2}$
(C) $0, \frac{8}{3}$
(D) $\frac{8}{3}, \frac{5}{2}$

(10) Determine k such that the quadratic equation $x^2 + 7(3 + 2k) - 2x(1 + 3k) = 0$ has equal roots :

(A) 2,7
(B) 7,5
(C) $2, -\frac{10}{9}$
(D) None of these

(11) Discriminant of the roots of the equation $-3x^2 + 2x - 8 = 0$ is

(A) -92
(B) -29
(C) 39
(D) 49

(12) The nature of the roots of the equation $x^2 - 5x + 7 = 0$ is

(A) No real roots
(B) 1 real root
(C) Can't be determined
(D) None of these

(13) Find the value of k such that the sum of the square of the roots of the quadratic equation $x^2 - 8x + k = 0$ is 40 :

(A) 12
(B) 2
(C) 5
(D) 8

(14) Find the value of p for which the quadratic equation $x^2 + p(4x + p - 1) + 2 = 0$ has equal roots :

(A) $-1, \frac{2}{3}$
(B) 3,5
(C) $1, -\frac{4}{3}$
(D) $\frac{4}{3}, 2$

(15) The length of a hypotenuse of a right triangle exceeds the length of its base by $2\ \mathrm{cm}$ and exceeds twice the length of the altitude by $1\ \mathrm{cm}$. Find the length of each side of the triangle (in cm) :

(A) 6,810
(B) $7,24,25$
(C) $8,15,17$
(D) $7,40,41$

(16) A two-digit number is such that the product of its digits is 12. When 9 is added to the number, the digits interchange their places, find the number:

(A) 62
(B) 34
(C) 26
(D) 43

(17) A plane left 40 minutes late due to bad weather and in order to reach its destination, $1600\ \mathrm{km}$ away in time, it had to increase its speed by $400\ \mathrm{km/h}$ from its usual speed. Find the usual speed of the plane :

(A) $600\ \mathrm{km/h}$
(B) $750\ \mathrm{km/h}$
(C) $800\ \mathrm{km/h}$
(D) None of these

(18) A shopkeeper buys a number of books for Rs. 80. If he had bought 4 more for the same amount, each book would have cost Re. 1 less. How many books did he buy?

(A) 8
(B) 36
(C) 24
(D) 28

(19) The squares have sides $x\ \mathrm{cm}$ and $(x+4)\mathrm{cm}$. The sum of their areas is $656\ \mathrm{cm}^2$. find the sides of the square.

(A) $8\ \mathrm{cm}, 12\ \mathrm{cm}$
(B) $12\ \mathrm{cm}, 15\ \mathrm{cm}$
(C) $6\ \mathrm{cm}, 10\ \mathrm{cm}$
(D) $16\ \mathrm{cm}, 20\ \mathrm{cm}$

(20) The real values of a for which the quadratic equation $2x^2-(a^3+8a-1)x+a^2-4a=0$ possesses roots of opposite signs are given by:

(A) $a>6$
(B) $a>9$
(C) $0<a<4$
(D) $a<0$

Answer Key

OBJECTIVE TYPE QUESTIONS

(1)	(D)	**(11)**	(A)
(2)	(C)	**(12)**	(A)
(3)	(B)	**(13)**	(A)
(4)	(C)	**(14)**	(A)
(5)	(B)	**(15)**	(C)
(6)	(B)	**(16)**	(B)
(7)	(B)	**(17)**	(C)
(8)	(C)	**(18)**	(B)
(9)	(D)	**(19)**	(D)
(10)	(C)	**(20)**	(C)