LIMIT

1. Definition

Let the function $y = f(x)$ be defined in a certain neighbourhood of a point $x = a$. The function $y = f(x)$ approaches the limit L ($y \to L$) as x approaches a ($x \to a$), if for every positive number h arbitrarily small, we are able to indicate $k > 0$ such that for all x, different from a and satisfying the inequality.

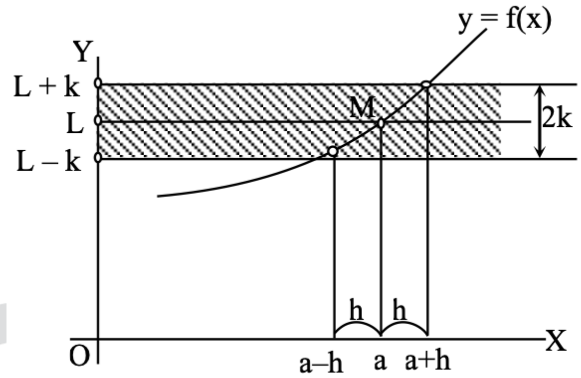
$| x - a | < h$

we have the inequality

$| f(x) - L | < k$

then $\lim\limits_{x \to a} f(x) = L$

or $f(x) \to L$ as $x \to a$.



If $f(x) \to L$ as $x \to a$, this is illustrated on the graph of the function $y = f(x)$ as above. Since from the inequality $|x - a| < h$ there follows the inequality $|f(x) - L| < k$, this means that for all points x that are not more distant from the point a than h, the points M of the graph of the function $y = f(x)$ lies within a band of width 2k bonded by the lines $y = L - k$ and $y = L + k$.

2. Indeterminate forms

Sometimes we come across some functions which do not have definite value corresponding to some particular value of the variable.

For example for the function $f(x) = \dfrac{x^2 - 4}{x - 2}$, $f(2) = \dfrac{4 - 4}{2 - 2} = \dfrac{0}{0}$ cannot be determined. Such a form is called an Indeterminate form.

Some Indeterminate forms are: $0/0, \infty/\infty, \infty - \infty, 0 \times \infty, 1^\infty, \infty^0, 0^0$

3. Existence of limit

Let f(x) has the indeterminate form at x = a then the limit of a function at a point a exists only when its left-hand limit and right-hand limit at that point are equal and unique. Thus $\lim\limits_{x \to a} f(x)$ exists

$\Rightarrow \lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x) =$ unique

i.e., RHL = LHL = unique value

4. Theorems on limit

The following theorems are very helpful for evaluation of limits:

(i) $\lim\limits_{x \to a} [k\, f(x)] = k \lim\limits_{x \to a} f(x)$, where k is a constant

(ii) $\lim\limits_{x \to a} [f(x) + g(x)] = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)$

(iii) $\lim\limits_{x \to a} [f(x) - g(x)] = \lim\limits_{x \to a} f(x) - \lim\limits_{x \to a} g(x)$

(iv) $\lim_{x\to a} [f(x).\, g(x)] = \lim_{x\to a} f(x).\ \lim_{x\to a} g(x)$

(v) $\lim_{x\to a} [f(x)/g(x)] = \lim_{x\to a} f(x) / [\lim_{x\to a} g(x)]$ (provided $\lim_{x\to a} g(x) \neq 0$)

(vi) $\lim_{x\to a} f[g(x)] = f[\lim_{x\to a} g(x)]$

(vii) $\lim_{x\to a} [f(x)+k] = \lim_{x\to a} f(x) + k$ where k is a constant

(viii) $\lim_{x\to a} \log\{f(x)\} = \log\{\lim_{x\to a} f(x)\}$

(ix) $\lim_{x\to a} [f(x)]^{g(x)} = \{\lim_{x\to a} f(x)\}^{\lim_{x\to a} g(x)}$

(x) $\lim_{x\to\pm\infty} f(x) = \lim_{x\to 0} f(1/x)$

5. Sandwich theorem or Squeeze playtheorems

If $g(x) \le f(x) \le h(x)$ and $\lim g(x) = \lim h(x)$ then $\lim f(x) = \lim g(x)$ or $\lim h(x)$

6. Some standard approaches to find limit of a function

6.1 Substitution

6.2 Factorization

6.3 Rationalization

6.4 By application of standard limits

6.1 Substitution: These are few important substitutions which are often used.

6.2 Factorization: Consider $\lim_{x\to a} \frac{f(x)}{g(x)}$. Factorise numerator and denominator and cancel out the common factor (x – a). After cancelling out the common factor (x – a), put x = a in given expression and the process is repeated till a meaningful number is obtained.

6.3 Rationalisation method:
In this method we rationalize the factor containing the square root and simplify and we put the value of x.

6.4. Application of Standard Limits: (Some standard limits)

(i) $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{x}{\sin x} = \lim_{x\to 0} \frac{\sin^{-1} x}{x} = 1$

$\lim_{x\to 0} \frac{x}{\sin^{-1} x} = 1;\ \lim_{x\to 0} \sin x = 0$

(ii) $\lim_{x\to 0} \cos x = \lim_{x\to 0} \left(\frac{1}{\cos x}\right) = 1$

(iii) $\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{x}{\tan x} = 1$

(iv) $\lim_{x\to\infty} \left(1+\frac{a}{x}\right)^x = \lim_{x\to 0} (1+ax)^{1/x} = e^a$

$$\lim_{x\to 0}(1+x)^{a/x} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{ax} = e^a$$

(v) $\lim_{x\to 0}\frac{a^x - 1}{x} = \log_e a \ (a > 0)$

(vi) $\lim_{x\to 0}\frac{e^x - 1}{x} = 1$

(vii) $\lim_{x\to a}\frac{x^n - a^n}{x - a} = n\,a^{n-1}, \ \lim_{x\to 1}\frac{x^n - 1}{x - 1} = n$

(viii) $\lim_{x\to 0}\frac{\ell n(1+x)}{x} = 1, \ \lim_{x\to 0}\frac{\log_a(1+x)}{x} = \log_a e$

(ix) $\lim_{x\to 0}\frac{(1+x)^n - 1}{x} = n,$

(x) $\lim_{x\to\infty}\frac{\sin x}{x} = \lim_{x\to\infty}\frac{\cos x}{x} = 0$

(xi) $\lim_{x\to\infty}\frac{\sin 1/x}{1/x} = 1$

(xii) $\lim_{x\to\infty} a^x = \begin{cases} 0 & \text{if } 0 \le a < 1 \\ 1 & \text{if } a = 1 \\ \infty & \text{if } a > 1 \\ \text{does not exist} & \text{if } a < 0 \end{cases}$

(xiii) $\lim_{x\to a}[f(x)]^{\phi(x)}$ = (provided $\lim_{x\to a}\phi$ is infinite & $\lim_{x\to a} f = 1$) = $e^{[\text{Lim}\,\phi\,\log f\,]}$

(xiv) $\lim_{x\to\infty}\frac{a_0x^m + a_1x^{m-1} + \ldots..}{b_0x^n + b_1x^{n-1} + \ldots...} = \frac{a_0}{b_0}$, for m = n

= 0, when m < n

= ∞ when m > n and $a_0b_0 > 0$

= –∞ when m > n and $a_0b_0 < 0$

7. Some limits which do not exist

(i) $\lim_{x\to 0}\left(\frac{1}{x}\right)$

(ii) $\lim_{x\to 0} x^{1/x}$

(iii) $\lim_{x\to 0}\frac{|x|}{x}$

(iv) $\lim_{x\to a}\frac{|x-a|}{x-a}$

(v) $\lim_{x\to 0}\sin 1/x$

(vi) $\lim_{x\to 0}\cos 1/x$

(vii) $\lim_{x\to 0} e^{1/x}$

(viii) $\lim_{x\to\infty}\sin x$

(ix) $\lim_{x\to\infty}\cos x$

SOLVED EXAMPLE

Ex.1 Evaluate $\underset{x\to 1}{\text{Lt}}\left[\frac{(2x-3)(\sqrt{x}-1)}{2x^2+x-3}\right]$

Sol. $\underset{x\to 1}{\text{Lt}}\left[\frac{(2x-3)(\sqrt{x}-1)}{2x^2+x-3}\right]$; $\frac{0}{0}$ form

$= \underset{x\to 1}{\text{Lt}}\left[\frac{(2x-3)(\sqrt{x}-1)}{(2x+3)(x-1)}\right]$

$= \underset{x\to 1}{\text{Lt}}\left[\frac{(2x-3)}{(2x+3)(\sqrt{x}+1)}\right] = \frac{-1}{(5)(2)} = -\frac{1}{10}$ **Ans.**

Ex.2 Evaluate $\underset{x\to\infty}{\text{Lt}}\frac{x^2+x+1}{3x^2+2x-5}$

Sol. $\underset{x\to\infty}{\text{Lt}}\frac{x^2+x+1}{3x^2+2x-5}$, $\left(\frac{\infty}{\infty}\text{form}\right)$

Put $x = \frac{1}{y}$

Limit $= \underset{y\to 0}{\text{Lim}}\frac{1+y+y^2}{3+2y-5y^2} = \frac{1}{3}$ **Ans.**

Shortcut for objective approach

Limit $= \frac{\text{Coeff. of highest power}}{\text{Coeff. of lowest power}} = \frac{1}{3}$ **Ans.**

Ex.3 Evaluate $\left[\frac{2\left\{\sqrt{3}\sin\left(\frac{\pi}{6}+h\right)-\cos\left(\frac{\pi}{6}+h\right)\right\}}{h\sqrt{3}\left(\sqrt{3}\cosh-\sinh\right)}\right]$

Sol. $\left[\frac{2\left\{\sqrt{3}\sin\left(\frac{\pi}{6}+h\right)-\cos\left(\frac{\pi}{6}+h\right)\right\}}{h\sqrt{3}\left(\sqrt{3}\cosh-\sinh\right)}\right]$; $\frac{0}{0}$ form

$= \left[\frac{2\left\{\frac{\sqrt{3}}{2}\sin\left(\frac{\pi}{6}+h\right)-\frac{1}{2}\cos\left(\frac{\pi}{6}+h\right)\right\}}{h\sqrt{3}\left(\frac{\sqrt{3}}{2}\cosh-\frac{1}{2}\sinh\right)}\right] = \left[\frac{-2\cos\left(\frac{\pi}{2}+h\right)}{h\sqrt{3}\cos\left(\frac{\pi}{6}+h\right)}\right] = \left[\frac{2\sinh}{h\sqrt{3}\cos\left(\frac{\pi}{6}+h\right)}\right]$

$= \underset{h\to 0}{\text{Lt}}\left(\frac{2}{\sqrt{3}}\right)\left(\frac{\sinh}{h}\right)\frac{1}{\cos\left(\frac{\pi}{6}+h\right)}$

$$= \frac{2}{\sqrt{3}} .(1) . \frac{1}{\cos\frac{\pi}{6}} = \frac{4}{3}$$ **Ans.**

Ex.4 Evaluate $\underset{x\to\frac{\pi}{2}}{\text{Lt}}$ (sec x– tan x)

Sol. $\underset{x\to\frac{\pi}{2}}{\text{Lt}}$ (sec x– tan x) ; ($\infty - \infty$ form)

$$= \underset{x\to\frac{\pi}{2}}{\text{Lt}} \left(\frac{1-\sin x}{\cos x}\right); \text{(now in } \frac{0}{0} \text{ form¬)}$$

Put $x = \left(\frac{\pi}{2} + h\right)$

$$\therefore \text{ Limit} = \underset{h\to 0}{\text{Lt}} \left[\frac{1-\sin\left(\frac{\pi}{2}+h\right)}{\cos\left(\frac{\pi}{2}+h\right)}\right] = \underset{h\to 0}{\text{Lt}} \left[\frac{1-\cosh}{-\sinh}\right]$$

$$= \underset{h\to 0}{\text{Lt}} \left[\frac{2\sin^2\frac{h}{2}}{-2\sin\frac{h}{2}\cos\frac{h}{2}}\right] = \underset{h\to 0}{\text{Lt}} \left[\frac{\sin\frac{h}{2}}{-\cos\frac{h}{2}}\right] = 0$$

Ex.5 Evaluate $\underset{h\to 0}{\text{Lt}} \left[\frac{\sqrt{x+h}-\sqrt{x}}{h}\right]$

Sol. $\underset{h\to 0}{\text{Lt}} \left[\frac{\sqrt{x+h}-\sqrt{x}}{h}\right]$; ($\frac{0}{0}$ form)

$$\text{Limit} = \underset{h\to 0}{\text{Lt}} = \frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \underset{h\to 0}{\text{Lt}} \frac{(x+h)-x}{(\sqrt{x+h}+\sqrt{x})}$$

$$= \underset{h\to 0}{\text{Lt}} \left[\frac{1}{\sqrt{x+h}+\sqrt{x}}\right] = \left[\frac{1}{2\sqrt{x}}\right]$$ **Ans.**

Ex.6 Evaluate = $\underset{x\to 0}{\text{Lt}} \left[\frac{x2^x - x}{1-\cos x}\right]$

Sol. Limit = $\underset{x\to 0}{\text{Lim}} \frac{x(2^x-1)}{1-\cos x}$; ($\frac{0}{0}$ form)

$$= \underset{x\to 0}{\text{Lim}} \frac{\left(\frac{2^x-1}{x}\right)}{\left(\frac{1-\cos x}{x^2}\right)} = \frac{\underset{x\to 0}{\text{Lim}}\left(\frac{2^x-1}{x}\right)}{\underset{x\to 0}{\text{Lim}}\frac{1}{2}\left(\frac{\sin^2 x/2}{x^2/4}\right)}$$

= 2 log 2 **Ans.**

Ex.7 Solve $\underset{x\to\frac{\pi}{2}}{\text{Lt}} (\sin x)^{\tan x}$

Sol. Let $u = \underset{x\to\frac{\pi}{2}}{\text{Lt}} (\sin x)^{\tan x}$

$\therefore \log u = \underset{x\to\frac{\pi}{2}}{\text{Lt}} [\tan x \log (\sin x)]$; ($\infty$. 0 form)

$= \underset{x\to\frac{\pi}{2}}{\text{Lt}} \left[\frac{\log \sin x}{\cot x}\right]$; ($\frac{0}{0}$ form)

$= \underset{x\to\frac{\pi}{2}}{\text{Lt}} \left[\frac{1}{\sin x}.\frac{\cos x}{(-\text{cosec}^2 x)}\right]$,

using L' Hospital Rule

$= \underset{x\to\frac{\pi}{2}}{\text{Lt}} (-\sin x \cos x) = 0 \quad \Rightarrow u = 1$ **Ans.**

ALTER :

Limit $= e^{\text{Lim}\phi(f-1)} = e^{\text{Lim} \tan x (\sin x - 1)}$

$= e^{\text{Lim}\frac{-\sin x \cos^2 x}{\cos x(1+\sin x)}} = e^0 = 1$ **Ans.**

Ex.8 If $f(x) = \sqrt{\frac{x - \sin x}{x + \cos^2 x}}$; find $\underset{x\to\infty}{\text{Lt}} f(x)$

Sol. $\underset{x\to\infty}{\text{Lt}} \sqrt{\frac{x - \sin x}{x + \cos^2 x}}$; (it is in $\frac{\infty}{\infty}$ form)

$= \underset{x\to\infty}{\text{Lt}} \sqrt{\frac{1 - \left(\frac{\sin x}{x}\right)}{1 + \left(\frac{\cos x}{x}\right)\cos x}} = \sqrt{\frac{1-0}{1+0.k}}$,

where $-1 \le k \le +1 \qquad = 1$ **Ans.**

Ex.9 Find $\underset{m\to\infty}{\text{Lim}} \left(\cos\frac{x}{m}\right)^m$

Sol. Limit $= e^{\underset{m\to\infty}{\text{Lim}} m\left(\cos\frac{x}{m} - 1\right)}$

$= e^{\underset{m\to\infty}{\text{Lim}} -\left(\frac{2x \sin^2\frac{x}{2m}}{\frac{x}{m}}\right)} = e^{\underset{m\to\infty}{\text{Lim}}\left(-2x.\sin\frac{x}{m}\right)}$

$= e^0 = 1$ **Ans.**

Ex.10 Evaluate $\underset{n\to\infty}{\text{Lt}} \left(\frac{\sin\frac{a}{n}}{\tan\frac{b}{n+1}}\right)$

Sol. Limit

$$= \underset{n\to\infty}{\text{Lt}} \left(\frac{\sin \frac{a}{n}}{\frac{a}{n}} \right) \left(\frac{\frac{b}{n+1}}{\tan \frac{b}{n+1}} \right) \cdot \frac{a}{b} \left(\frac{n+1}{n} \right)$$

$$= (1)\,(1)\,\frac{a}{b} \underset{n\to\infty}{\text{Lt}} \left(\frac{n+1}{n} \right) = \frac{a}{b}$$ **Ans.**

EXERCISE

Q.1 If $f(x) = \begin{cases} x^2+1 & , \quad x \geq 1 \\ 3x-1 & , \quad x < 1 \end{cases}$ then the value of $\lim_{x\to 1} f(x)$ is-

(A) 1 (B) 2

(C) 3 (D) Does not exist

Q.2 If $f(x) = 3 + \frac{1}{1+7^{1/(1-x)}}$ then-

(A) $\lim_{x\to 1^+} f(x) = 3$ (B) $\lim_{x\to 1^-} f(x) = 4$

(C) $\lim_{x\to 1} f(x) = 4$ (D) $\lim_{x\to 1} f(x)$ does not exist

Q.3 $\lim_{\theta\to -\frac{\pi}{4}} \frac{\cos\theta + \sin\theta}{\theta + \frac{\pi}{4}} =$

(A) $\sqrt{2}$ (B) 1

(C) 2 (D) Does not exist

Q.4 $\lim_{x\to\infty} \frac{x^3+x^2+1}{x^3-x^2+1} =$

(A) 0 (B) 1

(C) 2 (D) Does not exist

Q.5 $\lim_{x\to\infty} \frac{x^4+x^2+1}{x^5+x^2-1} =$

(A) 0 (B) 1

(C) 2 (D) Does not exist

Q.6 $\lim_{x\to\infty} \frac{\sqrt{3}x^5+x^2+13}{x^4+7x^2-\sqrt{17}} =$

(A) 0 (B) 2

(C) infinite (D) None

Q.7 $\lim_{x\to -1} \frac{x^3-2x-1}{x^5-2x-1} =$

(A) 2/3 (B) 1/3 (C) 4/3 (D) 5/3

Q.8 $\lim_{x\to\frac{\pi}{6}} \frac{2\sin^2 x+\sin x-1}{2\sin^2 x-3\sin x+1} =$

(A) 0 (B) 3 (C) –3 (D) 1

Q.9 $\lim_{x\to 3} \frac{x^3-7x^2+15x-9}{x^4-5x^3+27x-27} =$

(A) $\frac{2}{3}$ (B) $\frac{2}{9}$ (C) $\frac{1}{9}$ (D) 1

Q.10 $\lim_{x\to\infty}\left(\sqrt{x^2+1}-\sqrt{x^2-1}\right) =$

(A) 0 (B) 1 (C) 2 (D) None

Q.11 $\lim_{x\to 0}\dfrac{\sqrt{(1+x^2)}-\sqrt{(1+x)}}{\sqrt{(1+x^3)}-\sqrt{(1+x)}} =$

(A) 0 (B) 1
(C) 2 (D) 4

Q.12 If $\lim_{n\to\infty}\left(an-\dfrac{1+n^2}{1+n}\right)$ = b, a finite number then the ordered pair (a, b) is-

(A) (1, 1) (B) (–1, 1)
(C) (1, –1) (D) None of these

Q.13 $\lim_{x\to 2a^+}\dfrac{\sqrt{x-2a}+\sqrt{x}-\sqrt{2a}}{\sqrt{x^2-4a^2}}$, $a > 0 =$

(A) $\sqrt{2a}$ (B) $2\sqrt{a}$
(C) $1/2\sqrt{a}$ (D) $\sqrt{a}$

Q.14 The value of $\lim_{x\to\pi/4}\dfrac{\sqrt{1-\sqrt{\sin 2x}}}{\pi-4x} =$

(A) $-\dfrac{1}{4}$ (B) $\dfrac{1}{4}$

(C) $\dfrac{1}{2}$ (D) None

Q.15 $\lim_{x\to\infty}\left(\cos\sqrt{x+1}-\cos\sqrt{x}\right) =$

(A) 0 (B) 1
(C) 2 (C) None of these.

Q.16 $\lim_{x\to 2}\dfrac{(\cos\alpha)^x+(\sin\alpha)^x-1}{x-2}$, $x \in (0, \pi/2)$

(A) $\sin^2\alpha\ \ell n(\sin\alpha)$
(B) $\cos^2\alpha\ \ell n(\cos\alpha)$
(C) $\cos^2\alpha\ \ell n(\cos\alpha) - \sin^2\alpha\ \ell n(\sin\alpha)$
(D) $\cos^2\alpha\ \ell n(\cos\alpha) + \sin^2\alpha\ \ell n(\sin\alpha)$

Q.17 $\lim_{x\to\infty}\sqrt{x^2+x+1}-\sqrt{x^2+1} =$

(A) $\dfrac{2}{3}$ (B) 1

(C) $\dfrac{1}{2}$ (D) None

Q.18 $\lim_{x\to 0} \frac{\cos(\sin x) - \cos x}{x^4} =$

(A) 1 (B) 6

(C) $-\frac{1}{6}$ (D) $\frac{1}{6}$

Q.19 $\lim_{x\to a^-} \sqrt{a^2 - x^2}\ \cot\left(\frac{\pi}{2}\sqrt{\frac{a-x}{a+x}}\right)$ is-

(A) $\frac{a}{\pi}$ (B) $\frac{2a}{\pi}$ (C) $-\frac{a}{\pi}$ (D) $\frac{4a}{\pi}$

Q.20 The value of $\lim_{x\to 0}\left(1-\frac{1}{2^x}\right)\left(\frac{1}{\sqrt{\tan x + 4} - 2}\right)$

is-

(A) $\log_a 16$ (B) Does not exist

(C) 3 ln 2 (D) 4 ln 2

ANSWER KEY

EXERCISE

Q.No.	1	2	3	4	5	6	7	8	9	10
Ans.	B	D	A	B	A	C	B	C	B	A
Q.No.	11	12	13	14	15	16	17	18	19	20
Ans.	B	A	C	D	A	D	C	D	D	D