POLYNOMIALS

Concepts Covered

- Introduction to Polynomials, Polynomials in One Variable, Zeroes of a Polynomial, Value of Polynomial, Remainder Theorem, Factorization of Polynomials, Algebraic Identities

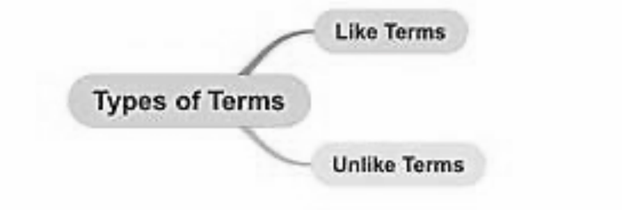
Basic Concepts and Important Terms:

(1) Constant: It is a symbol whose value always remains the same, whatever the situation be.
For Example: $8, -6, \pi, \frac{3}{8}, \frac{7}{15}$ etc.

(2) Variable: A symbol which may be assigned different numerical values i.e. changes according to the situation.
For Example: x, y, z, ax; $2x^2 - 7x + 5$; here, x is variable.
Similarly, area of circle $= \pi r^2$; here, π is a constant and r is variable.

(3) Algebraic Expressions: A combination of constants and variables, connected by some or all of the operations $+, -, x$ and $\div$ is known as algebraic expression.
For Example: $5x - 2y, 7x - 3y + 5z \ldots\ldots$ etc.

(4) Terms of an Algebraic Expression: The various parts of an algebraic expression that are separated by 't' or '-' sign are called terms.
For Example: In the expression $3x + 4y - 7$, we call $3x, 4y$ and -7 as terms.
There are two types of terms present in the polynomial.



Like Terms:

Terms having the same literal coefficients are called like terms.
For Example:
(i) $15x^2, -19x^2$ and $35x^2$ are all like terms.
(ii) $8x^2y, 5x^2y$ and $-7x^2y$ are all like terms.
Unlike Terms:
Terms having different literal coefficients are called unlike terms.
For Example: $5x^2, -10x$ and $15x^3$ are unlike terms.

(5) Coefficient: Any factor of a term is called the coefficient of the remaining term.
For Example:
(i) In $\mathbf{8x}, \mathbf{8}$ is coefficient of $\mathbf{x}$ **and** $\mathbf{x}$ is coefficient of 8.
(ii) In $\mathbf{2x^4} - \mathbf{5x^3} + \frac{2}{3}\mathbf{x^2} - \sqrt{2}\mathbf{x} + \mathbf{7}$, cofficient of $\mathbf{x^4}, \mathbf{x^3}, \mathbf{x^2}$ and $\mathbf{x}$ are $\mathbf{2}, -\mathbf{5}$, $\frac{2}{3}$ and $-\sqrt{\mathbf{2}}$, respectively, while 7 is the constant term.

(6) Exponent: The exponent of a number tells how many times a number should be used in multiplication.
In polynomials, an exponent is always a whole number $(0,1,2,3,\ldots.)$.
For Example: $2x^2 - 2x + 3$
In the given example, exponent of x in the first term is 2. Consider the examples:
(i) $\frac{7}{x} = 7x^{-1}$. Here, the exponent of x is -1, which is not a whole number. Therefore, this expression is not a polynomial.
(ii) $3\sqrt{z} = 3z^{\frac{1}{2}}$. Here, the exponent of z is $\frac{1}{2}$, which is not a whole number. Therefore, this expression is not a polynomial.

Degree of a Polynomial in One Variable:

The highest index of the variable in a polynomial of one variable is called the degree of the polynomial.
For Example:
(i) $11x^3 - 7x^2 + 5x + 2$ is a polynomial of degree 3.
(ii) $15x^6 - 8x + 7$ is a polynomial of degree 6.

Degree of polynomial in Two or More Variables:

The sum of the exponents of the variables in each term is taken up and the highest sum so obtained is called the degree of a polynomial in two or more variables.
For Example:
(i) $x^1y^5 - \frac{1}{5}x^2y^3 + x^1y^1 - \frac{1}{3}$
The given polynomial has two variables x and y.

- $\mathbf{x^1y^5}$, the sum of exponents = 6
- $\mathbf{\frac{1}{5}x^2y^3}$, the sum of exponents = 5
- $\mathbf{x^1y^1}$, the sum of exponents = 2
- $\mathbf{\frac{1}{3}}$, the sum of exponents = 0

Hence, the degree of polynomial is 6.

(ii) $4a^2b^4c^1 - 3a^1b^2c^2 + 2a^1b^1c^1 - 7$
The given polynomial has three variables a, b, and c.

- $\mathbf{4a^2b^4c^1}$, the sum of exponents = 7
- $\mathbf{3a^1b^2c^2}$, the sum of exponents = 5
- $\mathbf{2a^1b^1c^1}$, the sum of exponents = 3
- **7**, the sum of exponents = 0

Hence, the degree of polynomial is 7.

Example:

Write the degree of the following polynomials:
(i) $\mathbf{p(x) = 2x^3 - x + \frac{1}{\sqrt{5}}}$
(ii) $\mathbf{p(g) = 6g^5 - \frac{2}{5}g^2 + g - \frac{1}{3}}$
(iii) $\mathbf{p(x) = 5x^2 + 2x + 5}$
Solution : (i) $p(x) = 2x^3 - x + \frac{1}{\sqrt{5}}$
Since, the highest power of x in $p(x)$ is 3, the degree of polynomial $2x^3 - x + \frac{1}{\sqrt{5}}$ is 3.
(ii) $p(g) = 6g^5 - \frac{2}{5}g^2 + g - \frac{1}{3}$
As the highest power of g in the polynomial $p(g)$ is 5, the degree of polynomial $6\ g^5 - \frac{2}{5}\ g^2 + g - \frac{1}{3}$ is 5.
(iii) $p(x) = 5x^2 + 2x + 5$
As the highest power of x in $p(x)$ is 2, the degree of polynomial $p(x) = 5x^2 + 2x + 5$ is 2.

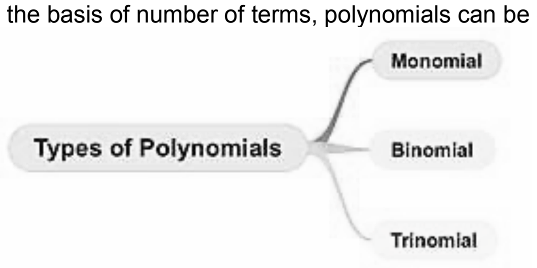
Check Your Concept - 1

Find the degree of the polynomial:
(i) $\frac{(y^3+2y+1)}{7} - \frac{9}{2}y^2 - y^6$
(ii) $-7x^2y^2 + 9x^3y^2 - 4x^2y^5$
(iii) $(2x + 3)(3x - 1)$

Classification of Polynomials

(1) Classification of polynomials on the basis of number of terms

On the basis of number of terms, polynomials can be classified as:



(i) Monomial:

An algebraic expression that contains only one term is called a monomial.
For Example: $2x, 3, 4t, 9pq, 2c$ etc... Each of these expressions contains only one term and hence is called monomial.

(ii) Binomial:

An algebraic expression that contains only two unlike terms is called a binomial.
For Example:
(i) $6x + 2x^2$
The given expression is a binomial since it contains two unlike terms, that is, $6x$ and $2x^2$ which are separated by a plus sign.
(ii) $6x - 2x^2$
The given expression is a binomial since it contains two unlike terms, that is, $6x$ and $2x^2$ which are separated by a minus sign.
If the terms, $6x$ and $2x^2$ are separated by a multiplication sign $(6x \times 2x^2)$, then it is not binomial because when we multiply two unlike terms, a monomial is formed. $6x \times 2x^2 = (6 \times 2)(x \times x^2) = 12x^3$

(iii) Trinomial:

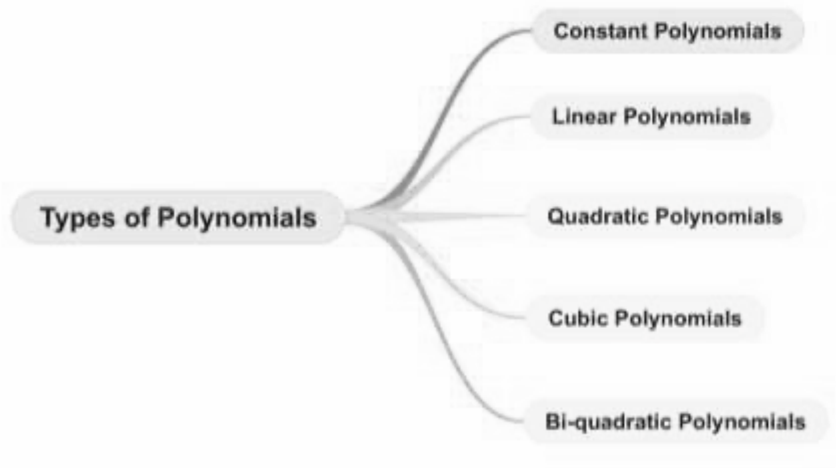
An algebraic expression that contains only three unlike terms is called a trinomial.
For Example: (i) $x + x^2 - 3x^3$ is trinomial, since it contains three unlike terms namely, x, x^2 and $-3x^3$.
(ii) $7y^3 + 8y^2 + 9y$ is trinomial, since it contains three unlike terms namely, $9y, 8y^2$ and $7y^3$.

Example:

State which of these is monomial, binomial and trinomial?
(i) $5 \times a + a$
(ii) $7a^2 + 8b + 9c$
(iii) $6a^2 + 5b^2$
Solution: (i) $5 \times a + a = 5a + a = 6a$
Here, number of terms is 1. Hence, it is a monomial.
(ii) $7a^2 + 8b + 9c$
Here, number of terms is 3. Hence, it is a trinomial.
(iii) $6a^2 + 5b^2$
Here, number of terms is 2. Hence, it is a binomial.

(2) Classification of polynomials on the basis of degree of variables

On the basis of degree of variable, polynomials are classified as



(i) Constant polynomial:

A polynomial of degree zero is called a constant polynomial.
For Example:
(i) $f(x) = 2, f(x) = 2x^0$
(ii) $g(x) = \frac{1}{2}, g(x) = \frac{1}{2}x^0$
The given polynomial has degree zero. Therefore, it is called a constant polynomial.

(ii) Linear polynomial:

A polynomial of degree one is called a linear polynomial. Linear polynomial may be a monomial or a binomial.
For Example:

(i) $y + 2$
(ii) $4x$
$\mathbf{y + 2}$ is a binomial, $\mathbf{4x}$ is a monomial. The highest degree of the given polynomials is 1; therefore, they are called linear polynomials.

(iii) Quadratic polynomial:

A polynomial of degree two is a called a quadratic polynomial. Quadratic polynomial may be a monomial or a binomial or a trinomial.

For Example:
(i) $2x^2$
(ii) $2x^2 + 3$
(iii) $2x^2 - 3x + 4$
In the above examples $2x^2$ is a monomial, $2x^2 + 3$ is a binomial, and $2x^2 - 3x + 4$ is a trinomial.
The highest degree of the given polynomials is 2 therefore, they are called quadratic polynomials.

(iv) Cubic polynomial:

A polynomial of degree three is called cubic polynomial.

For Example:
(i) $4x^3 + 2x^2 - 3x + 4$
(ii) $12x^3 - 3x + 48$
In the given examples, the highest degree of the polynomials is 3, therefore, they called are cubic polynomials.

(v) Biquadratic polynomial:

A polynomial of degree four is called a biquadratic polynomial.

For Example:
(i) $10x^4 + 4x^3 + 3x^2 - 8x - 12$
In the first example, only one variable x is present and the highest degree of the polynomials is 4; hence, it is called biquadratic polynomial.
(ii) $12x^2y^2 - 3x^2y^1 + 2x^1y^1 + 48$
In the second polynomial, two variables x and y are there, in which the sum of the powers of the variables in each term is taken up and the highest sum so obtained is the degree of polynomial. Here, the highest degree of the polynomial is $[2 + 2 = 4]$; hence, it is a called biquadratic polynomial.

Example:

Identify the type of the polynomials given below (on the basis of degree):

(i) $5y + 3$ **(ii) $\sqrt{3} + x^2 + x$** **(iii) $y^3 + y^2 + y + 1$**

Solution : (i) $5y + 3$
In this polynomial, the highest power of y is 1 , so it is a linear polynomial
(ii) $\sqrt{3} + x^2 + x$
In this polynomial, the highest power of x is 2, so it is a quadratic polynomial
(iii) $y^3 + y^2 + y + 1$
In this polynomial, the highest power of y is 3, so it is a cubic polynomial.

General Equation of Polynomial of n^{th} Degree

An algebraic expression of the form : $\mathbf{p(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots\ldots\ldots\ldots + a_1x^1 + a_0x^0}$ where $a_n \neq 0$ and $a_0, a_1, a_2, \ldots\ldots\ldots\ldots a_n$ are real numbers and each power of x is a positive integer, is called a polynomial.

Note: The exponent of variable in each term of a polynomial non-negative integer.

Value of a Polynomial

If p(x) is a polynomial in variable x and α is any real number, the value obtained by replacing x by α in p(x) is called value of p(x) at x = α and is denoted by p(x).

Example:

Find the value of :
(i) p(x) = $x^3 - 6x^2 + 11x - 6$ (at $x = -2$)
(ii) $q(y) = 2y^3 - 2y + \sqrt{10}$ at $y = 2$
(iii) $p(r) = 4r^2 - 2r + 6$ at $r = a$

Solution : (i) p(–2) = $(-2)^3 - 6(-2)^2 + 11(-2) - 6 = -8 - 24 - 22 - 6 \Rightarrow p(-2) = -60$
(ii) $q(y) = 2y^3 - 2y + \sqrt{10}$ at $y = 2$
$q(2) = 2(2)^3 - 2(2) + \sqrt{10} = 2 \times 8 - 4 + \sqrt{10} = 16 - 4 + \sqrt{10}$
$q(2) = 12 + \sqrt{10}$, which is the required value of $q(y)$ at 2
(iii) $p(r) = 4r^2 - 2r + 6$ at $r = a$
$p(a) = 4(a)^2 - 2(a) + 6$
$p(a) = 4a^2 - 2a + 6$, which is the required value of $p(r)$ at $r = a$

Zero of a Polynomial

A real number α is a root or zero of a polynomial if and only if $f(\alpha) = 0$.
So, if $f(x)$ is a polynomial and α is a real number, then the following are equivalent:

- **α is a zero of $f(x)$.**
- **$x = \alpha$ is a solution of $f(x) = 0$.**
- **$x - \alpha$ is a factor of $f(x)$.**

Example:

Find the zero of polynomial $3x + 6$.
Solution: Given polynomial $p(x) = 3x + 6$
On putting $p(x) = 0$, we get $3x + 6 = 0$
$3x = 0 - 6$
$3x = -6$
$x = \frac{-6}{3} = -2$
Hence, $x = -2$ i.e., $p(-2) = 0$ is the zero of polynomial, $3x + 6$.

Example:

If p(x) = $x^3 - 6x^2 + 11x - 6$, find the zero of polynomial.
Solution: p(1) = $1^3 - 6(1)^2 + 11(1) - 6$ = 1 – 6 + 11 – 6 = 0
p(2) = $(2)^3 - 6(2)^2 + 11(2) - 6$ = 8 – 24 + 22 – 6 = 0
p(3) = $(3)^3 - 6(3)^2 + 11(3) - 6$ = 27 – 54 + 33 – 6 = 0
Thus, 1, 2 and 3 are called the zero of polynomial p(x).

Example:

Check whether -2 and 2 are zeros of the polynomial $t^2 - t - 6$.
Solution: On putting $t = -2$ in equation (1) we get,
$p(t) = t^2 - t - 6$
$p(-2) = (-2)^2 - (-2) - 6$
$p(-2) = 4 + 2 - 6$
$p(-2) = 6 - 6$
$p(-2) = 0$
Again, on putting $t = 2$ in equation (1) we get,
$p(t) = t^2 - t - 6$
$p(2) = (2)^2 - (2) - 6$
$p(2) = 4 - 2 - 6$
$p(2) = 2 - 6$
$p(2) = -4$
Therefore, -2 is a zero of the polynomials $t^2 - t - 6$, but 2 is not as the value of $p(2)$ is not equal to zero.

Remainder Theorem

Let $p(x)$ be any polynomial of degree ≥ 1 and ' a ' be any real number. If $p(x)$ is divided by $(x - a)$, the remainder is equal to $p(a)$.
Proof: Let $q(x)$ be the quotient and $r(x)$ be the remainder when $p(x)$ is divided by $(x - a)$ then
$p(x) = (x - a)q(x) + r(x)$
where $r(x) = 0$ or degree of $r(x) <$ degree of $(x - a)$.
but $(x - a)$ is a polynomial of degree ' 1 ' and the polynomial of degree less than one is a constant.
∴ either $r(x) = 0$ or $r(x) =$ constant
Let $r(x) = r$, then
$p(x) = (x - a)q(x) + r$
Putting $x = a$ then
$p(a) = (a - a)q(a) + r$
$\Rightarrow\ p(a) = r$
∴ When $p(x)$ is divided by $(x - a)$ then the remainder is $p(a)$.
Note:
(1) If a polynomial $p(x)$ is divided by $(x + a)$, the remainder value of $p(x)$ at $x = -a$ is $p(-a)$.
(2) If a polynomial $p(x)$ is divided by $(ax - b)$, the remainder is the value of $p(x)$ at $x = b/a$. i.e., $p(b/a)$.
(3) If a polynomial $p(x)$ is divided by $(ax + b)$, the value of remainder at $x = -b/a$ is $p(-b/a)$.
(4) If a polynomial $p(x)$ is divided by $(b - ax)$, the remainder value is equal to $p(b/a)$ at $x = b/a$.

Example:

Find the remainder when $x^3 + 3x^2 + 3x + 1$ is divided by $x + 1$.
Solution: $x + 1 = 0 \Rightarrow x = -1$
∴ Remainder $= p(-1) = (-1)^3 + 3(-1)^2 + 3(-1) + 1 = -1 + 3 - 3 + 1 = 0$

Example:

Find the value of k, if $x^{21} + 2x^{20} + 3x + k$ is divisible by $x + 1$.

Solution: The polynomial $x^{21} + 2x^{20} + 3x + k$ is divided by $x + 1$. Therefore, the remainder is equal to -1.

Putting $x = -1$ in the given polynomial, we get

$p(-1) = 0$

$p(-1) = (-1)^{21} + 2(-1)^{20} + 3(-1) + k$

$p(-1) = -1 + 2 - 3 + k$

$p(-1) = -4 + 2 + k \Rightarrow p(-1) = -2 + k$

$p(-1) = k - 2$

We know that $p(-1) = 0$

$k - 2 = 0 \Rightarrow k = 2$

Example:

Find the remainder when the polynomial $p(x) = 4x^3 - 12x^2 + 14x - 3$ is divided by $g(x) = x - \frac{1}{2}$.

Solution: By remainder theorem,

We know that when the polynomial, $p(x)$ is divided by $g(x) = \left(x - \frac{1}{2}\right)$ then the remainder is equal to $p\left(\frac{1}{2}\right)$

$p(x) = 4x^3 - 12x^2 + 14x - 3$

$p\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)3 - 12\left(\frac{1}{2}\right)2 + +14\left(\frac{1}{2}\right) - 3$

$p\left(\frac{1}{2}\right) = 4 \times \frac{1}{8} - 12 \times \frac{1}{4} + 14 \times \frac{1}{2} - 3$

$p\left(\frac{1}{2}\right) = \frac{4}{8} - \frac{12}{4} + \frac{14}{2} - 3 \Rightarrow p\left(\frac{1}{2}\right) = \frac{1}{2} - 3 + 7 - 3$

$p\left(\frac{1}{2}\right) = \frac{1}{2} + \frac{4}{1} - 3$

$p\left(\frac{1}{2}\right) = \frac{1 + 8}{2 \times 1} - 3$

$p\left(\frac{1}{2}\right) = \frac{9}{2} - 3 = \frac{3}{2}$

Hence, required remainder, $p\left(\frac{1}{2}\right) = \frac{3}{2}$

Factor Theorem:

Let $p(x)$ be a polynomial of degree ≥ 1 ' a ' be a real number such that $p(a) = 0$ then $(x - a)$ is a factor of $p(x)$. Conversely if $(x - a)$ is a factor of $p(x)$ then $p(a) = 0$.

Proof: Let $p(x)$ be a polynomial of degree ≥ 1 ' a ' be a real number such that $p(a) = 0$ then we have to show that $(x - a)$ is a factor of $p(x)$.

Let $q(x)$ be the coefficient when $p(x)$ is divided by $(x - a)$.

By remainder theorem, $p(x)$ when divided by $(x - a)$ gives the remainder equal to $p(a)$.

$\therefore p(x) = (x - a)q(x) + p(a)$

$\Rightarrow p(x) = (x - a)q(x)$

$\Rightarrow (x - a)$ is a factor of $p(x)$

Conversely,

Let $(x - a)$ be a factor of $p(x)$. Then we have to prove that $p(a) = 0$.

Now, $(x - a)$ is a factor of $p(x)$.

$\Rightarrow$ $p(x)$ when divided by $(x - a)$ gives remainder zero.

But by remainder theorem, $p(x)$ divided by $(x - a)$ giving the remainder equal to $p(a)$.

$\therefore$ $p(a) = 0$

Note:

(1) If $(x + a)$ is a factor of a polynomial $p(x)$ if $p(-a) = 0$.

(2) $(ax - b)$ is a factor of a polynomial $p(x)$ if $p\left(\frac{b}{a}\right) = 0$.

(3) $ax + b$ is a factor of a polynomial $p(x)$ if $p\left(-\frac{b}{a}\right) = 0$.

(4) $(x - a)(x - b)$ is a factor of a polynomial $p(x)$ if $p(a) = 0$ and $p(b) = 0$

Example:

Show that $2x - 3$ is a factor of the polynomial $2x^3 - 9x^2 + x + 12$.

Solution : Let $f(x) = 2x^3 - 9x^2 + x + 12$ be the polynomial.

By Factor Theorem, $2x - 3$ is a factor of polynomial $2x^3 - 9x^2 + x + 12$ if $f\left(\frac{3}{2}\right) =$

Now, $f(x) = 2x^3 - 9x^2 + x + 12$

$f\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^3 - 9\left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right) + 12$

$= \frac{54}{8} - \frac{81}{4} + \frac{3}{2} + 12$

$= \frac{27-81+6+48}{4} = \frac{81-81}{4}$

Hence, $2x-3$ is a factor of $2x^3-9x^2+x+12$.

Example:

If $x+1$ is a factor of $2x^3-ax^2+x+12$, find the value of a.

Solution: Let $f(x) = 2x^3-ax^2+x+12$ be the given polynomial.

By Factor Theorem, $x+1$ is a factor of $2x^3-ax^2+x+12$ if $f(-1)=0$.

Now,

$f(-1) = 2(-1)^3 - a(-1)^2 + (-1) + 12$

$0 = -2 - a - 1 + 12$

$a = 9$

$\therefore\ f(x) = 2x^3 - 9x^2 + x + 12$

Example:

Find the value of k, if $x+k$ is a factor of $x^3+kx^2-2x+k+4$.

Solution: Let $f(x) = x^3+kx^2-2x+k+4$ be the given polynomial.

By Factor Theorem, $x+k$ is a factor of $x^3+kx^2-2x+k+4$ if $f(-k)=0$.

Now,

$f(-k) = (-k)^3 + k(-k)^2 - 2(-k) + k + 4$

$0 = -k^3 + k^3 + 2k + k + 4$

$0 = 3k + 4$

$k = -\frac{4}{3}$

Hence, $x+k$ is a factor of $f(x)$, if $k = -\frac{4}{3}$

Check Your Concept - 2

(i) Find the remainder when the polynomial
(a) $f(a) = a^3 - 3a^2 + 4a + 50$ is divided by $(a+3)$.
(b) $f(x) = x^3 + 3x^2 + 3x + 1$ is divided by $5+2x$.
(ii) Using the remainder theorem, find the remainder when f(x) is divided by g(x).
(a) $f(x) = x^3 + 4x^2 - 3x + 10, \quad g(x) = x +$
(b) $f(x) = 9x^3 - 3x^2 + x - 5, \quad g(x) = x - \frac{2}{3}$
(iii) Determine which of the following polynomial has $(x+1)$ as a factor:
(a) $x^3 - x^2 - (2+\sqrt{2})x - \sqrt{2}$
(b) $x^4 + 3x^3 + 3x^2 + x + 1$
(iv) If $p(x) = x^3 - 3x + 2$, evaluate $p(2) - p(-1) + p(\frac{1}{2})$.
(v) Find the value of k for which $(x+k)$ is a factor of the polynomial $f(x) = x^3 + kx^2 - 2x + k + 6$.
(vi) If x + 1 and x – 1 are factors of $mx^3 + x^2 - 2x + n$, find the value of m and n.

Factorisation

Long Division Method

Step 1: First, arrange the terms of the dividend and the divisor in the descending order of their degrees.
Step 2: Now, the first term of the quotient is obtained by dividing the first term of the dividend by the first term of the divisor.
Step 3: Then, multiply all the terms of the divisor by the first term of the quotient and subtract the result from the dividend.
Step 4: Consider the remainder as the new dividend and proceed as before.
Step 5: Repeat this process till we obtain a remainder which is either 0 or a polynomial of degree less than that of the divisor.

Example:

Divide $2x^3+9x^2+4x-15$ by $2x+5$

Solution: $2x+5)\overline{2x^3+9x^2+4x-15}(x^2+2x-3$

$2x^3+5x^2$
$\underline{(-)\ (-)}$
$4x^2+4x$
$4x^2+10x$
$\underline{(-)(-)}$
$-6x-15$
$-6x-15$
$\underline{(+)(+)}$
0

$\therefore (2x^3+9x^2+4x-15) \div (2x+5) = x^2+2x-3$.

Application of Factor Theorem in Factorization of Polynomials

Algorithm

Step 1: Obtain the given polynomial $f(x)$.
Step 2: Obtain the constant term in $f(x)$ and find all of its possible factors, $a_1, a_2 \dots$..
Step 3: Take one of the factors and replace x by it in the given polynomial. If the polynomial reduces to zero, then $(x - a_1)$ is a factor of the polynomial; combine this procedure till you get as many factors as the degree of the polynomial.
Step 4: Put $f(x) = k(x - a_1)(x - a_2)$, where k is a constant.
Step 5: Substitute any values of x other than $a_1, a_2 \dots \dots$ in the equation obtained in step IV and get the value of k.
Step 6: Substitute the value of k in $f(x) = k(x - a_1)(x - a_2)$.

Example:

Using factor theorem, factorize the polynomial $x^4 + x^3 - 7x^2 - x + 6$.
Solution: Let $f(x) = x^4 + x^3 - 7x^2 - x + 6$

Factors of constant term in $f(x)$ is
$\pm1, \pm2, \pm3, \pm6$
$f(1) = 1 + 1 - 7 - 1 + 6 = 0$
$\Rightarrow (x - 1)$ is a factor of $f(x)$
$f(-1) = 1 - 1 - 7 + 1 + 6 = 0$
$\Rightarrow (x + 1)$ is also a factor of $f(x)$
$f(2) = 2^4 + 2^3 - 7(2)^2 - (2) + 6 = 0$
$\Rightarrow (x - 2)$ is a factor
Also $f(-2) = (-2)^4 + (-2)^3 - 7(-2)^2 - (-2) + 6 \neq 0$
$\Rightarrow (x + 2)$ is a not a factor of $f(x)$
$f(-3) = (-3)^4 + (-3)^3 - 7(-3)^2 - (-3) + 6 = 0$
$\Rightarrow (x + 3)$ is a factor of $f(x)$

Since $f(x)$ is a polynomial of degree 4, it cannot have more than 4 factors.
$\therefore f(x) = k(x - 1)(x + 1)(x - 2)(x + 3) \dots .(1)$
Now, let $x = 0$
$\Rightarrow f(0) = 6 = k(-1)(1)(-2)(3)$
$\Rightarrow 6 = 6k \Rightarrow k = 1$
Substituting in equation (1) we get
$x^4 + x^3 - 7x^2 - x + 6 = (x - 1)(x + 1)(x - 2)(x + 3)$

Factorization by Splitting the Middle Term

To factorize a quadratic polynomial, we split the coefficient of x into two numbers such that their product is equal to the product of coefficient of x^2 and the constant term and sum is equal to the coefficient of x.
Factorization of polynomials of the form $ax^2 + bx + c$:
Step 1: Take the product of the constant term and the coefficient of x^2, i.e., ac.
Step 2: Now, this product ac is to split into two factors m and n such that m + n is equal to the coefficient of x, i.e., b.
Step 3: Then, we pair one of them, say mx, with ax^2 and the other nx, with c and factorize.

Example:

Factorise: $6x^2 - x - 22$
Solution: Coefficient of $x = -1$

Product of coefficient of x^2 and constant term $= 6(-22) = -132$
i.e, the sum of two numbers $= -1$
and product $= -132$
$\therefore$ The numbers $= -12, 11$
$6x^2 - x - 22 = 6x^2 - 12x + 11x - 22 = 6x(x - 2) + 11(x - 2)$
$(x - 2)(6x + 11)$

Example:

Factorise: $6x^2 + 19x + 15$.
Solution : Here, $6 \times 15 = 90 = 10 \times 9$ and $10 + 9 = 19$

$\therefore 6x^2 + 19x + 15$
$= 6x^2 + 10x + 9x + 15$
$= 2x(3x + 5) + 3(3x + 5) = (2x + 3)(3x + 5)$

Algebraic Identities:

(1) $x^2 + 2xa + a^2 = (x + a)^2$
(2) $x^2 - 2xa + a^2 = (x - a)^2$
(3) $x^2 - a^2 = (x + a)(x - a)$
(4) $x^3 - a^3 = (x - a)(x^2 + xa + a^2)$

(5) $x^3 + a^3 = (x + a)(x^2 - xa + a^2)$
(6) $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$
(7) $x^3 + y^3 + z^3 = 3xyz$ if $x + y + z = 0$
(8) $x^3 + 3x^2a + 3xa^2 + a^3 = (x + a)^3$
(9) $x^3 - 3x^2a + 3xa^2 - a^3 = (x - a)^3$

Example:

Factorize $\mathbf{8x^3 + 64y^3}$.
Solution: $8x^3 + 64y^3 = (2x)^3 + (4y)^3$
$= (2x + 4y)[(2x)^2 - (2x)(4y) + (4y)^2]$
$= 2(x + y)(4x^2 - 8xy + 16y^2)$
$\therefore$ Factors of $8x^3 + 64y^3 = 2(x + y)(4x^2 - 8xy + 16y^2)$.

Example:

Factorize $\mathbf{(x - y)^3 + (y - z)^3 + (z - x)^3}$.
Solution: Let $x - y = a, y - z = b; z - x = c$.
then $a + b + c = x - y + y - z + z - x = 0$
$[\because a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$
and if $a + b + c = 0 \Rightarrow a^3 + b^3 + c^3 = 3abc\,]$
$\Rightarrow (x - y)^3 + (y - z)^3 + (z - x)^3 = 3(x - y)(y - z)(z - x)$

Check Your Concept - 3

(i) Factorize the following algebraic expressions:
(a) $(a + b)^2 - 5a - 5b - 6$
(b) $x^3 - y^3 - x + y$
(c) $a^9 - b^9$
(d) $6x^2 + 7x - 3$
(ii) Without actual division, prove that $2x^4 - 5x^3 + 2x^2 - x + 2$ is divisible by $x^2 - 3x + 2$
(iii) If $x^2 + \frac{1}{x^2} = 27$, find the value of $x - \frac{1}{x}$.

(1) Find the remainder if $(x^3 - ax^2 + 2x - a)$ is divided by $(x - a)$.

Solution: $f(x) = (x^3 - ax^2 + 2x - a)$

Now, a = 0

x = a

Now, $f(a) = a^3 - a\,a^2 + 2a - a$

$= a^3 - a^3 + 2a - a = a$

The required remainder is a.

(2) Find if $(x - 2)$ is a factor of $(x^3 - 8)$.

Solution: $f(x) = (x^3 - 8)$

Here, $f(2) = (2)^3 - 8$

$= 8 - 8 = 0$

Therefore, $(x - 2)$ is a factor of $(x^3 - 8)$.

(3) Find the value of (k+20) for which $(x - 1)$ is a factor of $(2x^3 + 9x^2 + x + k)$.

Solution: $f(x) = (2x^3 + 9x^2 + x + k)$

$x - 1 = 0$

$\Rightarrow x = 1$

$f(1) = 2 \times 1^3 + 9 \times 1^2 + 1 + k$

$= 2 + 9 + 1 + k = 12 + k$

Given that $(x - 1)$ is a factor of f(x).

$\therefore f(1) = 0.$

$f(1) = 12 + k = 0$

$k = -12$

$k + 20 = 8$

(4) The polynomials $(2x^3 + x^2 - ax + 2)$ and $(2x^3 - 3x^2 - 3x + a)$ when divided by $(x - 2)$ leave the same remainder. Find the value of a.

Solution: $p(x) = 2x^3 + x^2 - ax + 2$

$p(x) = 2x^3 + x^2 - ax + 2$

Putting x = 2 in p(x),

we get

$p(2) = 22 - 2a$

$q(x) = 2x^3 - 3x^2 - 3x + a$

Putting x = 2 in q(x),

we get

$q(2) = a - 2$

Remainder are same, So,

a - 2 = 22 - 2a

3a = 24

a = 8

(5) Factorize: $(ax + by)^2 + (bx - ay)^2$.

Solution: $(ax + by)^2 + (bx - ay)^2$

$= a^2 x^2 + b^2 y^2 + 2abxy + b^2 x^2 + a^2 y^2 - 2abxy$

$= a^2 x^2 + b^2 y^2 + b^2 x^2 + a^2 y^2$

$= a^2 x^2 + b^2 x^2 + b^2 y^2 + a^2 y^2$

$= x^2 (a^2 + b^2) + y^2(a^2 + b^2)$

$= (a^2 + b^2)(x^2 + y^2)$

(6) The value of $\frac{(27)^{n/3} \times (8)^{-n/6}}{162^{-n/2}}$ is equal to

Solution: Bring all the numbers in the lowest prime numbers as bases,

$$= \frac{3^{3\times n/3} \times 2^{3\times -n/6}}{3^{4\times -n/2} \times 2^{-n/2}}$$

$$= \frac{3^n \times 2^{-n/2}}{3^{-2n} \times 2^{-n/2}} = 3^{3n}$$

(7) If $(x^3 + ax^2 + bx + 6)$ has $(x - 2)$ as a factor and leaves a remainder 3 when divided by $(x - 3)$, find the value of $b - a$.

Solution: Let $f(x) = (x^3 + ax^2 + bx + 6)$

So, $f(3) = 3^3 + a \times 3^2 + b \times 3 + 6 = 3$

$= 27 + 9a + 3b + 6 = 3$

$= 9a + 3b + 33 = 3$

$= 9a + 3b = 3 - 33$

$= 9a + 3b = -30$

$= 3a + b = -10$ ….(i)

Given that $(x - 2)$ is a factor of f(x).

By the Factor Theorem, (x – a) will be a factor of f(x) if f(a) = 0 and therefore
f(2) = 0
$f(2) = 2^3 + a \times 2^2 + b \times 2 + 6 = 0$
= 8 + 4a+ 2b + 6 = 0
= 4a + 2b = −14
= 2a + b = −7 ….(ii)
Subtracting (ii) from (i), we get,
a = −3
Substituting the value of a = -3 in (i),
we get,
3(−3) + b = −10
−9 + b = −10
b = −10 + 9 ⇒ b = −1
a = −3 and b = − 1
b − a = 2

(8) Find the remainder if $(2x^4 + 6x^3 + 2x^2 + x – 8)$ is divided by (x+3).

Solution: $f(x) = (2x^4 + 6x^3 + 2x^2 + x – 8)$
Now, x + 3 = 0
x = −3
We know that when f(x) is divided by
(x + 3) the remainder is f(−3).
$f(−3) = 2(−3)^4 + 6(−3)^3 + 2(−3)^2 – 3 – 8$
= 162 – 162 + 18 – 3 – 8
= 18 – 11= 7
The required remainder is 7.

(9) Without actual division, show that $(x^3 – 3x^2 – 13x + 15)$ is exactly divisible by $(x^2 + 2x – 3)$.

Solution: Let $f(x) = x^3 – 3x^2 – 13x + 15$
Now, $x^2 + 2x – 3 = x^2 + 3x – x – 3$
= x (x + 3) – 1 (x + 3) ⇒ (x + 3) (x – 1)
Thus, f(x) will be exactly divisible by $x^2 + 2x – 3 = (x + 3)(x – 1)$
if (x + 3) and (x – 1) are both factors of f(x), so by factor theorem,
we should have f(-3) = 0 and f(1) = 0.
Now, $f(-3) = (-3)^3 – 3(-3)^2 – 13(-3) + 15$
= – 27 – 3 x 9 + 39 + 15 ⇒ –27 – 27 + 39 + 15
= –54 + 54 = 0
And $f(1) = 1^3 – 3 \times 1^2 – 13 \times 1 + 15$
= 1 – 3 – 13 + 15 ⇒ 15 – 15 = 0
f(-3) = 0 and f(1) = 0
So, $x^2 + 2x – 3$ divides f(x) exactly.

(10) If $x - \frac{1}{x} = 8$. Find $x^2 + \frac{1}{x^2}$.

Solution: $\left(x - \frac{1}{x}\right) = 8 \Rightarrow \left(x - \frac{1}{x}\right)^2 = 8^2 \Rightarrow x^2 + \frac{1}{x^2} - 2.x.\frac{1}{x} = 64$

$\Rightarrow x^2 + \frac{1}{x^2} - 2 = 64$

$\Rightarrow x^2 + \frac{1}{x^2} = 64 + 2 \therefore x^2 + \frac{1}{x^2} = 66$

(11) If $x - \frac{1}{x} = 2$, find $x^4 + \frac{1}{x^4}$.

Solution: $x - \frac{1}{x} = 2 \Rightarrow \left(x - \frac{1}{x}\right)^2 = 2^2$

$\Rightarrow x^2 + \frac{1}{x^2} - 2.x.\frac{1}{x} = 4$

$\Rightarrow x^2 + \frac{1}{x^2} = 6 \Rightarrow \left(x^2 + \frac{1}{x^2}\right)^2 = 6^2$

$\Rightarrow (x^2)^2 + \left(\frac{1}{x^2}\right)^2 + 2.x^2.\frac{1}{x^2} = 36$

$\Rightarrow x^4 + \frac{1}{x^4} + 2 = 36$

$\Rightarrow x^4 + \frac{1}{x^4} = 34$

(12) If x − y = 15 and xy = 30, find the value of $x^3 - y^3$.

Solution: $(x - y)^3 = x^3 - y^3 - 3xy(x - y)$
$\Rightarrow (15)^3 = x^3 - y^3 - 3 \times 30 \times 15$
$\Rightarrow 3375 = x^3 - y^3 - 1350$
$\Rightarrow x^3 - y^3 = 3375 + 1350$
$\Rightarrow x^3 - y^3 = 4725$

(13) Find the value of a^2 for which (x + a) is a factor of the polynomial $f(x) = x^3 + ax^2 - 2x + a + 6$.

Solution: (x + a) is a factor of $f(x) = x^3 + ax^2 - 2x + a + 6$

$f(-a) = 0$

$(-a)^3 + a(-a)^2 - 2(-a) + a + 6 = 0$

$3a = -6$

$a = -2 \Rightarrow a^2 = 4$

(14) If $x = 3 + \sqrt{8}$, find the value of $x^2 + \frac{1}{x^2}$.

Solution: $x^2 + \frac{1}{x^2}$,

Given, $x = 3 + \sqrt{8}$

$\therefore \frac{1}{x} = \frac{1}{3+\sqrt{8}} \times \frac{3-\sqrt{8}}{3-\sqrt{8}} = 3 - \sqrt{8}$

Hence, $x + \frac{1}{x} = 3 + \sqrt{8} + 3 - \sqrt{8} = 6$

$x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2$

$= 6^2 - 2 = 34$

OBJECTIVE TYPE QUESTION

(1) What is the remainder when $(4x^3 - 3x^2 + 2x - 1)$ is divided by $(x + 2)$?
(A) 49 (B) 48
(C) – 49 (D) None

(2) The remainder when $4a^3 - 12a^2 - 14a - 3$ is divided by $2a - 1$ is:
(A) $\frac{5}{3}$ (B) $\frac{3}{2}$
(C) 0 (D) $\frac{-1}{2}$

(3) If $(x^3 + ax^2 + bx + 6)$ has $(x - 2)$ as a factor and leaves a remainder 3 when divided by $(x - 3)$, the values of 'a' and 'b' are:
(A) $a = -3, b = -1$ (B) $a = 3, b = -1$
(C) $a = -3, b = 1$ (D) $a = 3, b = 1$

(4) If $4x^4 - 3x^3 - 3x^2 + x - 7$ is divided by $1 - 2x$, the remainder will be
(A) $\frac{57}{8}$ (B) $-\frac{59}{8}$
(C) $\frac{55}{8}$ (D) $-\frac{55}{8}$

(5) The polynomials $ax^3 + 3x^2 - 3$ and $2x^3 - 5x + a$ when divided by $(x - 4)$ leaves remainders R_1 and R_2, respectively, then value of a if $2R_1 - R_2 = 0$.
(A) $-\frac{18}{127}$ (B) $\frac{18}{127}$
(C) $\frac{17}{127}$ (D) None

(6) If $(x - a)$ is a factor of $x^6 - ax^5 + x^4 - ax^3 + 3x - a + 2$, then value of a is
(A) 1 (B) 2
(C) –2 (D) –1

(7) A quadratic polynomial is exactly divisible by $(x + 1)$ and $(x + 2)$ and leaves the remainder 4 after division $(x + 3)$, the polynomial is
(A) $x^2 + 6x + 4$ (B) $2x^2 + 6x + 4$
(C) $2x^2 + 6x - 4$ (D) $x^2 + 6x - 4$

(8) The values of a and b so that the polynomial $x^3 - ax^2 - 13x + b$ is divisible by $(x - 1)$ & $(x + 3)$ are
(A) a = 15, b = 3 (B) a = 3, b = 15
(C) a = – 3, b=15 (D) a = 3, b = –15

(9) The product of all the solutions of $x^4 - 16 = 0$, is
(A) –16 (B) –4
(C) 4 (D) 16

(10) $x - 8xy^3 =$
(A) $x(1 - 2y)(1 + 2y + 4y^2)$ (B) $x(1 + 2y)(1 + 2y + 4y^2)$
(C) $x(1 - 2y)(1 - 2y + 4y^2)$ (D) $x(1 + 2y)(1 - 2y + 4y^2)$

(11) Factors of $x^4 - x^2 - 12$ is
(A) $(x + 2)(x - 2)(x^2 + 3)$ (B) $(x + 3)(x - 3)(x^2 + 2)$
(C) $(x + 2)(x - 2)(x^2 - 3)$ (D) $(x^2 + 2)(x^2 - 6)$

(12) The value of k for which $(x + 2)$ is a factor of $(x + 1)^7 + (3x + k)^3$ is
(A) –7 (B) 7
(C) –1 (D) $-6 - 3^{(7/3)}$

(13) $3\sqrt{3}x^3 + y^3 =$
(A) $(\sqrt{3}x + y)(3x^2 - \sqrt{3}xy + y^2)$ (B) $(\sqrt{3}x - y)(3x^2 + \sqrt{3}xy + y^2)$
(C) $(\sqrt{3}x + y)(3x^2 + \sqrt{3}xy + y^2)$ (D) $(\sqrt{3}x - y)(3x^2 - \sqrt{3}xy - y^2)$

(14) Factors of $x^3 + x^2 + x + 1 =$
(A) $(x + 1)(x^2 - 1)$ (B) $(x - 1)(x^2 + 1)$
(C) $(x - 1)(x^2 - 1)$ (D) $(x + 1)(x^2 + 1)$

(15) $y^3 - 2y^2 - y + 2 =$
(A) $(y - 2)(y - 1)(y + 1)$ (B) $(y + 2)(y - 1)(y + 1)$
(C) $(y - 2)(y + 1)^2$ (D) $(y + 1)(y - 1)^2$

(16) If $x^{\frac{1}{3}} + y^{\frac{1}{3}} + z^{\frac{1}{3}} = 0$, then
(A) $x^3 + y^3 + z^3 = 0$ (B) $x + y + z = 27xyz$
(C) $(x + y + z)^3 = 27xyz$ (D) $x^3 + y^3 + z^3 = 27xyz$

(17) $8a^3 - 2a^2b - 15ab^2 =$
(A) $(4a + 5b)(2a^2 - 3ab)$ (B) $(4a - 5b)(2a^2 - 3ab)$
(C) $(4a - 5b)(2a^2 + 3ab)$ (D) $(4a + 5b)(2a^2 + 3ab)$

(18) If $x + y + z = 0$, then $x^3 + y^3 + z^3$ is
(A) xyz (B) 2xyz
(C) 3xyz (D) zero

(19) If $x + 1/x = 15$, then $x^2 + 1/x^2$ is equal to
(A) 223 (B) 210
(C) 225 (D) 225 + 1/225

(20) $1 – x + x^2 – x^3 =$
(A) $(1 + x)(1 – x^2)$ (B) $(1 – x)(1 + x^2)$
(C) $(1 – x)(1 – x^2)$ (D) $(1 + x)(1 + x^2)$

Answer Key

CHECK YOUR CONCEPT

(1) **(i)** 6 **(ii)** 7 **(iii)** 2

(2) **(i)** (a) -16 ,(b) $-\frac{27}{8}$ **(ii)** (a) 22 , (b) -3 **(iii)** (a) Yes (b) No
(iv) $\frac{5}{8}$ **(v)** -2 **(vi)** m=2, n= -1

(3) **(i)** (a) $(a + b - 3)(a + b - 2)$ (b) $(x - y)(x^2 + xy + y^2 - 1)$
(c) $(a - b)(a^2 + ab + b^2)(a^6 + a^3b^3 + b^6)$ (d) $(2x + 3)(3x - 1)$
(iii) ± 5

OBJECTIVE TYPE QUESTION

(1)	(C)	**(6)**	(D)	**(11)**	(A)	**(16)**	(C)
(2)	(B)	**(7)**	(B)	**(12)**	(B)	**(17)**	(A)
(3)	(A)	**(8)**	(B)	**(13)**	(A)	**(18)**	(C)
(4)	(B)	**(9)**	(B)	**(14)**	(D)	**(19)**	(A)
(5)	(B)	**(10)**	(A)	**(15)**	(A)	**(20)**	(B)